

BOSON STARS WITH NEGATIVE COSMOLOGICAL CONSTANT

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Abstract

We consider boson star solutions in a D -dimensional, asymptotically anti-de Sitter spacetime and investigate the influence of the cosmological term on their properties. We find that for $D > 4$ the boson star properties are close to those in four dimensions with a vanishing cosmological constant. A different behavior is noticed for the solutions in the three dimensional case. We establish also the non-existence of static, spherically symmetric black holes with a harmonically time-dependent complex scalar field in any dimension greater than two.

1 INTRODUCTION

Recently a tremendous amount of interest has been focused on anti-de Sitter (AdS) spacetime. This interest is mainly motivated by the proposed correspondence between physical effects associated with gravitating fields propagating in AdS spacetime and those of a conformal field theory on the boundary of AdS spacetime [1, 2].

Being a maximally symmetric spacetime, AdS spacetime is also an excellent model to investigate questions of principle related to the quantisation of fields propagating on a curved background, the interaction with the gravitational field and issues related to the lack of global hyperbolicity.

In view of these developments, an examination of the classical solutions of gravitating fields in asymptotically AdS (AAdS) space seems appropriate.

Recently, some authors have discussed the properties of gravitating SU(2) nonabelian fields for $\Lambda < 0$ [3, 4, 5]. They obtained some surprising results, which are strikingly different from the results found in the asymptotically flat case (for example the existence of regular dyon solutions without a Higgs field). Insofar as the no-hair theorem is concerned, it has been shown that there exist stable black hole solutions in SU(2) Einstein-Yang-Mills (EYM) theory that are AAdS [3, 4]. However, the same system in the presence of a Higgs scalar presents solutions with very similar properties to the asymptotically-flat space counterparts [6, 7, 8]. More recently it has been shown that the $U(1)$ Higgs field equations have a vortex solution in four-dimensional AAdS [9]. Static spherically symmetric solutions of general relativity with $\Lambda < 0$, coupled to a real scalar field, have been studied in [10]. Surprisingly enough, the solutions have finite energy although there are naked singularities and present interesting properties in the context of the AdS/CFT correspondence.

All of these studies assume that the matter fields have the same symmetries as the underlying spacetime; in particular, the scalar fields are assumed to be invariant under the action of the stationary Killing vector. However, the consideration of static configurations in which a complex scalar field varies harmonically with time, with a static energy-momentum tensor, yields new kinds of regular configurations, the so-called 'boson star' solutions.

The study of boson stars can be traced back to the work of Kaup [11] and Ruffini and Bonazzalo [12] more than 30 years ago. They found asymptotically flat, spherically symmetric equilibrium solutions of the

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Einstein-Klein-Gordon equations. These boson stars are macroscopic quantum states and are only prevented from collapsing gravitationally by the Heisenberg uncertainty principle. Although boson stars have many similarities to their fermionic counterparts, there are many interesting differences. For example, boson stars also exhibit a critical mass and critical particle number. Later works considered the more complicated possibilities of the presence of a self interaction for the scalar field [13, 14, 15] or a nonminimal coupling of the scalar field to gravity [16]. Jetzer and van der Bij extended the model to include a $U(1)$ gauge charge [17] (see also [18]). Boson stars in the presence of a dilaton or an axidilaton have also been studied by various authors [19], as well as boson-fermion stars [20].

All these models have demonstrated the same characteristic: new interactions tends to increase the critical values of mass and particle number, although the particular values are very model dependent. The stability against perturbations around the equilibrium state has been discussed also by a number of authors [21]-[24]. An extensive review of the boson star properties is given in [25], [26].

All of these solutions assumed a spherically symmetric ansatz; a more recent development is the study of axially symmetric boson star configurations [27, 28, 29]. Asymptotically flat, rotating boson stars have been obtained in [30, 31].

The recent interest in these solitonic objects is largely due to the suggestion that the dark matter could be bosonic in nature. With scalar fields often used to model early universe, there is the possibility that such solutions could condense to form boson stars.

Most of the works on boson star properties have been carried out on the assumption that spacetime is asymptotically flat. Because of the physical importance of these objects, it is worthwhile to study generalizations in a different cosmological background. We may hope that similar to a nonabelian theory, the different asymptotic structure of the spacetime will affect the properties of the solutions leading to some new effects.

The goal of this paper is to reexamine the basic properties of soliton stars in the presence of a negative cosmological constant. The solutions we are looking for are the asymptotically AdS analogues of the Ruffini and Bonazzalo configurations [12]. To our knowledge, a discussion of this case is not available in the literature.

We find that the basic properties of a four dimensional soliton star will remain basically the same even in the presence of a negative cosmological constant. However a nonzero Λ term in the action implies a complicated power decay of the fields at infinity, rather than exponentially. Also, as expected, the parameter of the solutions found in Ref.[12] (mass, particle number, effective radius) remains no longer valid and new values are found for every value of Λ . The main effect of a nonzero cosmological constant is to decrease the maximal mass of the star.

Motivated by recent interest in higher dimensional gravity with negative cosmological constant, we look also for spherically symmetric boson stars with $\Lambda \leq 0$ in spacetimes with a number of dimensions $D \neq 4$. It is always of interest to see how the dimensionality of spacetime affects the physical consequences of a given theory.

The higher-dimensional boson star solutions might be relevant in the context of superstring theory as well as for understanding how the behaviour depends on the dimensionality of the spacetime. In particular we are interested in the five-dimensional case. To our knowledge very little is known about the properties of these type of configurations in $D \neq 4$, although the exact boson star solutions in three-dimensional gravity with $\Lambda < 0$ were found in [32]. A qualitative discussion of the boson star properties in the context of brane world models with compact large extra dimensions has been done in [33]. As argued there, the properties of large soliton stars will remain similar even in the context of the brane world and large extra dimensions scenarios. However, at distances smaller than the compactification radius, the properties of a boson stars are quite different than in the $(3+1)$ -dimensional case.

We find that for $D > 3$ and a negative cosmological constant, the properties of the solutions are similar to the well-known four-dimensional asymptotically flat case. The total mass parameter M rises quickly with the initial value of the scalar field at the origin φ_0 to a maximum, then drops and has several local minima and maxima until it approaches an asymptotic value for large φ_0 . A similar behaviour is found for the particle number N . The solutions for a vanishing cosmological constant are discussed as a particular case, since they present the same general pattern. A separate section is dedicated to the rather special case $D = 3$. Exact solutions in this case (in the limit of large self interaction) are known to exist. However, the properties of the boson star solutions in AdS_3 spacetime are different as compared to the higher dimensional case. We

notice the existence of a maximal allowed value for the central density and the absence of local extrema for N and M . Also, no solutions are found in the asymptotically flat limit.

One may ask about the possible relevance of these macroscopic quantum states within the AdS/CFT correspondence. Since the proposal of Maldacena's conjecture, which gives a correspondence between a theory of gravity in AdS_D spacetime and a field theory on its $(D-1)$ -dimensional boundary, much intense work has been devoted to get a deeper understanding of its implications. The AdS/CFT correspondence for a scalar field has been discussed by a number of authors, however without considering these type of time-dependent solutions. We present in this paper only a preliminary approach to this question. The boundary stress tensor and the associated conserved charges are computed for three, four and five dimensions. In three and five dimensions, the counterterm prescription [34] gives an additional vacuum (Casimir) energy, which agrees with that found in the context of AdS/CFT correspondence.

The paper is structured as follows: in Section 2 we present the general framework and analyse the field equations and boundary conditions exactly. In Section 3 we address the question of AAdS black hole solutions with complex scalar hair. In $4D$ asymptotically flat space, the boson stars do not allow for black hole horizon inside them [35]. We generalize this result to include a negative Λ , for any $D \geq 3$. In Section 4 the field equations are solved numerically. It is shown how the dimensionality of spacetime affects the properties of solutions and several cases are examined in detail. In section 5 the stability of the solutions is considered. The mass of the solutions and the boundary stress tensor are computed in Section 6. We conclude with Section 7 where the results are compiled.

2 GENERAL FRAMEWORK AND EQUATIONS OF MOTION

2.1 Basic ansatz

Evolution of a minimally coupled complex scalar field Φ in D dimensions is described by the action (throughout this paper we will use units in which $c = \hbar = 1$)

$$S_D = - \int_{\mathcal{M}} d^D x \left(\frac{\sqrt{-g_D}}{16\pi G_D} (R - 2\Lambda) + L_\Phi \right) - \frac{1}{8\pi G_D} \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{-h} K, \quad (1)$$

where the second term is the Hawking-Gibbons surface term [36], K is the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric of the boundary. Of course, this term does not affect the equations of motion but is important when discussing quantum properties of gravitating solutions.

In (1) the lagrangian density of a complex self-gravitating scalar field reads

$$L_\Phi = -\sqrt{-g_D} (g^{ij} \Phi_{,i}^* \Phi_{,j} + V(\Phi)), \quad (2)$$

where the asterisc denotes complex conjugate. In this paper we consider only the case $V(\Phi) = \mu^2 \Phi^* \Phi$, where μ is the scalar field mass, without including a scalar self-interaction term. As found in [14] for four dimensional asymptotically flat solutions, although the inclusion of a $\lambda|\Phi|^4$ term drastically changes the value of the maximum mass and the corresponding critical central density of the boson star solutions, the qualitative features are essentially similar to the non-self interaction case.

The Lagrangian density (2) is invariant under a global phase rotation $\Phi \rightarrow \Phi e^{-i\alpha}$; that implies the existence of a conserved current

$$J^k = i g^{kl} (\Phi_{,l}^* \Phi - \Phi_{,l} \Phi^*), \quad (3)$$

and an associated conserved charge, namely, the number of scalar particles

$$N = \int d^{D-1}x \sqrt{-g_D} J^t. \quad (4)$$

The field equations are obtained by varying the action (1) with respect to field variables g_{ij} and Φ

$$R_{ij} - \frac{1}{2} g_{ij} R + \Lambda g_{ij} = 8\pi G_D T_{ij}, \quad (5)$$

$$\nabla^2 \Phi - \mu^2 \Phi = 0, \quad (6)$$

where the energy momentum tensor is defined by

$$T_{ij} = \Phi_{,i}^* \Phi_{,j} + \Phi_{,j}^* \Phi_{,i} - g_{ij}(g^{km} \Phi_{,k}^* \Phi_{,m} + \mu^2 |\Phi|^2). \quad (7)$$

As usual with the scalar field, the Klein-Gordon equation is redundant.

In this paper we assume a stationary ansatz $\Phi = \phi(r)e^{-i\omega t}$ describing a spherically symmetric bound state of the scalar field with frequency ω . Hereafter, a prime denotes the derivative with respect to r and an overdot $\partial/\partial t$. As we assume spherical symmetry it is convenient to use the metric form

$$ds^2 = \frac{dr^2}{F(r)} + r^2 d\Omega_{D-2}^2 - F(r)e^{-2\delta(r)} dt^2, \quad (8)$$

where $d\Omega_{D-2}^2 = \omega_{ab} dx^a dx^b$ is the line element on a unit $(D-2)$ -dimensional sphere and

$$F(r) = 1 - \frac{2m(r)}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-1)}. \quad (9)$$

The function $m(r)$ is related to the local mass-energy density up to some D -dependent factor. Substituting in (7) we find for the nonvanishing components of the energy-momentum tensor,

$$\begin{aligned} T_t^t &= -\rho = -F\phi'^2 - \frac{\omega^2 e^{2\delta}}{F} \phi^2 - \mu^2 \phi^2, \\ T_r^r &= P_r = F\phi'^2 + \frac{\omega^2 e^{2\delta}}{F} \phi^2 - \mu^2 \phi^2, \end{aligned}$$

for the energy density and the radial pressure while for the $(D-2)$ angular components we find

$$T_\theta^\theta = P_t = -F\phi'^2 + \frac{\omega^2 e^{2\delta}}{F} \phi^2 - \mu^2 \phi^2.$$

We remark that the radial component of the pressure P_r and tangential components P_t are different, precluding a macroscopic description of these configurations in terms of an effective perfect fluid energy-momentum tensor.

2.2 Reduced action and virial relations

If one is only interested in spherically symmetric solutions it is much simpler to work with a reduced action where this symmetry of the spacetime is factored out. Expressing the curvature scalar R in terms of the metric function $m(r)$ and $\delta(r)$, we obtain the following expression for the effective action of our static spherically symmetric system

$$S(m, \delta, \phi) = \int_0^\infty \left(\frac{(D-2)}{8\pi G_D} m' e^{-\delta} - r^{D-2} e^{-\delta} \left(F\phi'^2 - \frac{e^{2\delta} \omega^2 \phi^2}{F} + V(\phi) \right) \right) dr. \quad (10)$$

This form of the reduced action allows to derive a useful virial relation by using a scaling technique. In this way it is possible to better understand the reason for the existence of nontrivial solutions and to provide nonexistence theorems.

We will use an approach proposed by Heusler in [37]. Let us assume the existence of a solution $m(r), \delta(r), \phi(r)$ with suitable boundary conditions at the origin and at infinity. Then each member of the 1-parameter family

$$m_\lambda(r) \equiv m(\lambda r), \quad \delta_\lambda(r) \equiv \delta(\lambda r), \quad \phi_\lambda(r) \equiv \phi(\lambda r) \quad (11)$$

assumes the same boundary values at $r = 0$ and $r = \infty$, and the action $S_\lambda \equiv S[m_\lambda, \delta_\lambda, \phi_\lambda]$ must have a critical point at $\lambda = 1$, $[dS/d\lambda]_{\lambda=1} = 0$. Thus we obtain the virial relation valid for a D -dimensional

spacetime

$$\int_0^\infty dr r^{D-2} e^{-\delta} \left((D-3) \left(1 + \frac{4m}{r^{D-3}} - \frac{2\Lambda r^2}{(D-2)(D-3)} \right) \phi'^2 + (D-1)V(\phi) - \left(D-1 - \frac{4(D-2)m}{r^{D-3}} - \frac{2(D-3)\Lambda r^2}{(D-1)(D-2)} \right) \frac{\omega^2 e^{2\delta} \phi^2}{F^2} \right) = 0. \quad (12)$$

By using this relation we can also discuss whether the cosmological constant can support a real scalar field. In asymptotically flat spacetimes, a well-known result implies the absence of scalar solitons. It can be seen from (12) that in any dimension $D \geq 3$ there are no static, spherically symmetric scalar solutions with negative cosmological constant and a positive potential, independently of whether or not gravity is taken into account.

Also, there are different proofs of the no-hair theorem for spherically symmetric scalar fields (see [37] for a set of references). The above scaling argument does not exclude black hole solutions with complex scalar hair (to this end, one has to replace the lower boundary in the action integral (10) by the horizon distance r_h). However, by using another sort of argument, we will prove in Section 3 the absence of such solutions, which clearly indicates the limits of scaling techniques.

For $V(\phi) = \mu^2 \phi^2$, the relation (12) can be simplified by using the Klein-Gordon equation written in the form

$$\frac{1}{2} (r^{D-2} e^{-\delta} F (\phi^2)')' = r^{D-2} \left(F \phi'^2 + \mu^2 \phi^2 - \phi^2 \frac{\omega^2 e^{2\delta}}{F} \right). \quad (13)$$

Integrating the above relation and making use of the boundary conditions from Section (2.3), we find

$$\int_0^\infty dr r^{D-2} e^{-\delta} \left(F \phi'^2 + \mu^2 \phi^2 - \phi^2 \frac{\omega^2 e^{2\delta}}{F} \right) = 0, \quad (14)$$

which, together with (12) gives the simple relation

$$\int_0^\infty dr r^{D-2} e^{-\delta} \left(\left(1 - \frac{m(3D-7)}{r^{D-3}} \right) \phi'^2 + \frac{\omega^2 e^{2\delta}}{F^2} \left(\frac{2\Lambda r^2}{(D-1)(D-2)} - \frac{m(D-3)}{r^{D-3}} \right) \phi^2 \right) = 0. \quad (15)$$

In the above relation, the factor in front of ϕ^2 has a negative sign. Thus, the existence of boson star solutions is supported by the factor $(1 - m(3D-7)/r^{D-3})$ in front of ϕ'^2 , which has no definite sign.

The relation (15) particularized for $D = 3$

$$\int_0^\infty dr r e^{-\delta} \left((1 - 2m) \phi'^2 + \frac{\omega^2 e^{2\delta} \Lambda r^2}{F^2} \phi^2 \right) = 0. \quad (16)$$

can be used to exclude the existence of three-dimensional asymptotically flat boson stars.

2.3 Field equations and boundary conditions

To perform numerical computations and order-of-magnitude estimations, it is useful to have a new set of dimensionless variables. Thus, we perform the rescalings $r \rightarrow r/\mu$, $\phi \rightarrow \sqrt{\frac{(D-2)}{16\pi G_D}} \phi$, $m \rightarrow m/\mu^{D-3}$ and $\Lambda \rightarrow \Lambda/\mu^2$, while the factor ω/μ is absorbed into the definition of the metric function δ .

Then we find the field equations

$$m' = \frac{r^{D-2}}{2} \left(F \phi'^2 + \phi^2 + \frac{e^{2\delta} \phi^2}{F} \right), \quad (17)$$

$$(e^{-\delta})' = r \left(e^{-\delta} \phi'^2 + e^{\delta} \frac{\phi^2}{F^2} \right), \quad (18)$$

$$(r^{D-2} e^{-\delta} F \phi')' = r^{D-2} e^{-\delta} \phi \left(1 - \frac{e^{2\delta}}{F} \right). \quad (19)$$

Taking into account the explicit form for the metric and the scalar field we find the particle number

$$N = \frac{4\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})} \int_0^\infty dr \phi^2 r^{D-2} \frac{e^\delta}{F}. \quad (20)$$

Similar to the $\Lambda = 0$ case, we can define a star radius

$$R = \frac{1}{N} \int d^{D-1}x \, r J^t \sqrt{-g_D} = \frac{4\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})N} \int_0^\infty dr \phi^2 r^{D-1} \frac{e^\delta}{F}. \quad (21)$$

Following the standard analysis, we can predict the boundary conditions and some general features of the finite-energy solutions. The boundary conditions for the system are the following. The non-singularity at the center of the star requires that $\phi(0) = \varphi_0$, $\phi'(0) = 0$, $m(0) = 0$. We find also that the boundary conditions at infinity obtained for an asymptotically flat spacetime remain valid, with $\phi = 0$ as the only acceptable value, and also that

$$\lim_{r \rightarrow \infty} \phi'^2 \sim O(1/r^{D+2+\epsilon}), \quad (22)$$

where ϵ is a small positive quantity.

The formal power series describing the above boundary conditions at $r = 0$ is

$$\phi(r) = \varphi_0 + \frac{\varphi_0(1 - e^{2\delta_0})}{2(D-1)} r^2 + O(r^3), \quad (23)$$

$$m(r) = \frac{\varphi_0^2(1 + e^{2\delta_0})}{2(D-1)} r^{D-1} + O(r^D), \quad (24)$$

$$\delta(r) = \delta_0 - \frac{1}{2} e^{2\delta_0} \varphi_0^2 r^2 + O(r^3). \quad (25)$$

The dimensionless value of the scalar field at the origin φ_0 is also used as a parametrization of the central density of the configuration. For $\Lambda = 0$, boson star solutions are characterized by an exponential decay of the scalar field, for which a mass term in the potential is responsible. The behaviour for $\Lambda < 0$ is different: the analysis of the field equations as $r \rightarrow \infty$ gives

$$\begin{aligned} \phi(r) &\sim \hat{\phi}_0 r^c + \dots, \\ m(r) &\sim M + \frac{(k^2 c^2 + 1) \hat{\phi}_0^2}{2(2c + D - 1)} r^{2c+D-1} + \dots, \\ e^{-\delta(r)} &\sim 1 + \frac{c \hat{\phi}_0^2}{2} r^{2c} + \dots, \end{aligned} \quad (26)$$

where $c = -\frac{1}{2} \left(D - 1 + \sqrt{(D-1)^2 + 4/k^2} \right)$, $k^2 = -2\Lambda/(D-2)(D-1)$ and $\hat{\phi}_0$ is a constant. Thus, the cosmological constant implies a complicated power decay at infinity, rather than an exponential one as found in an asymptotically flat space.

We will prove in Section (6) that the constant M in the above relations

is the *ADM* mass of a boson star up to a D -dependent factor. However, in the discussion of numerical solutions we will refer to M to as the mass of the solutions.

3 A NO-HAIR THEOREM

In general, when a theory allows regular configurations, it also allows black hole solutions. However, this is not the case for the action discussed in this paper. In four dimensions with $\Lambda = 0$, Peña and Sudarsky have proven that there are no black hole analogues of the regular boson star configurations [35]. Therefore, the collapse of a boson star results in a trivial Schwarzschild black hole. A crucial point in their proof was

the assumption of asymptotic flatness. However, in the last years it has been realized that the introduction of a (negative) cosmological constant may allow for configurations forbidden in the asymptotically flat case (see, for example, the existence of dyonic regular and black hole solutions in EYM theory with $\Lambda < 0$). The dimensionality of spacetime may also affect the existence and the properties of solutions in a given theory. A nontrivial example is the gravitating SU(2) theory, which presents spherically symmetric particle-like solutions in $D = 4$ [38] without counterparts in $D = 3$ [39] and $D = 5$ [40] dimensions.

In this section we prove the following straightforward generalization of the Peña-Sudarsky theorem for $D \geq 3$ and $\Lambda \leq 0$.

Theorem:

A D -dimensional, static, spherically symmetric, AAdS black hole spacetime, with regular event horizon, satisfying Einstein's equations with the matter fields fulfilling the weak energy condition (WEC) and its energy-momentum satisfying the condition

$$T_\theta^\theta \leq T_r^r, \quad (27)$$

is necessarily trivial (*i.e.* $T_i^j = 0$ and the only black hole is Schwarzschild-AdS spacetime).

Proof.: Since our arguments are very similar to [35], we present only the main steps. From the (rr) and (tt) Einstein equations we can derive the following relations

$$F' = \frac{16\pi r G_D}{D-2} T_t^t - \frac{2r\Lambda}{D-2} + \frac{D-3}{r}(1-F), \quad (28)$$

$$\delta' = \frac{8\pi r G_D}{(D-2)F} (T_t^t - T_r^r) \quad (29)$$

By using these relations and the conservation of the energy-momentum tensor $T_{i;k}^k = 0$ we find

$$e^\delta (e^{-\delta} T_r^r)' = \frac{1}{2Fr} \left[(D-3)(1-F - \frac{2\Lambda r^2}{(D-2)(D-3)})(T_t^t - T_r^r) + 2F(D-2)(T_\theta^\theta - T_r^r) \right]. \quad (30)$$

The WEC states that the energy density $\rho \equiv -T_t^t$ is semipositive definite and $T_r^r, T_\theta^\theta \geq T_t^t$. Thus, this condition and the assumption (27) implies that $e^{-\delta} T_r^r$ is a nonincreasing function of r and $e^{-\delta} T_r^r(r) \leq e^{-\delta} T_r^r(r_h)$. Here r_h is the event horizon radius, corresponding to a zero of the function F . However, by expressing (30) in terms of the proper radial distance $dx = \frac{dr}{\sqrt{F}}$ we find

$$e^\delta \frac{d}{dx} (e^{-\delta} T_r^r) = \frac{1}{2\sqrt{F}r} \left((D-3)(1 - \frac{2\Lambda r^2}{(D-2)(D-3)})(T_t^t - T_r^r) \right) \quad (31)$$

$$+ \frac{\sqrt{F}}{2r} (2(D-2)(T_\theta^\theta - T_r^r) - (D-3)(T_t^t - T_r^r)). \quad (32)$$

Since the left hand side of this equation is finite as $r \rightarrow r_h$, we find $T_r^r(r_h) = T_t^t(r_h) = -\rho(r_h) < 0$. However $e^{-\delta} T_r^r(\infty) = 0$, so we conclude $T_r^r(r) = 0$. The WEC plus condition (27) implies that $T_\theta^\theta = T_t^t = 0$ also. The only black hole solution of Einstein field equations is thus the Schwarzschild-AdS spacetime.

It can easily be seen that a complex scalar field with the lagrangian density (2) and $\Phi = \phi(r)e^{-i\omega t}$ satisfies the conditions of the theorem above as long as $V(\Phi) > 0$.

By using similar arguments we can also exclude the nonexistence of topological black holes. These are black holes with $\Lambda < 0$ for which the topology of the horizon is an arbitrary genus Riemann surface, the $(D-2)$ -sphere being replaced by a $(D-2)$ -dimensional space of negative or vanishing curvature (see [41] for a review).

Applying the same arguments in the case of a positive cosmological constant leaves open the question of the existence. We find only that the scalar field and the energy momentum tensor should vanish on the cosmological event horizon [42] while T_r^r is a strictly increasing quantity.

4 NUMERICAL SOLUTION

While an analytical solutions to the coupled nonlinear equations (17)-(19) appears to be intractable for every dimensions, the resulting system has to be solved numerically. In this way we find that the boson stars with negative cosmological constant may exist in any dimension $D \geq 3$.

A complete analysis of the complex correlation between the two parameters of the theory (φ_0, Λ) for a given spacetime dimension is beyond the purposes of this paper. To compare numerically the results with those known in the asymptotically flat case we focused on solutions with $D = 3, 4, 5$ and have varied the parameter φ_0 for a set of Λ .

For each choice of φ_0 , the eqs. (17)-(19) with the initial conditions (23) have a solution satisfying the boundary conditions (26) only when $e^\delta(r=0)$ takes on certain values. Different values of δ_0 corresponds to different number of nodes of the scalar field. We follow the usual approach and, by using a standard ordinary differential equation solver, we evaluate the initial conditions at $r = 10^{-3}$ for global tolerance 10^{-12} , adjusting for fixed shooting parameter and integrating towards $r \rightarrow \infty$. We solve the equations and find the functions m , δ and ϕ and also the values of M , N , R . The energy of these solutions is always concentrated in a small region.

Only the zero-node solutions will be given here. However, we found that similar to the $D = 4$, $\Lambda = 0$ case [13], ϕ can have nodes, giving rise to excited states.

4.1 Boson stars in four dimensions

We start by discussing the better known case $D = 4$. The behaviour of M , N as a function of $\phi(0)$ is well known for the $\Lambda = 0$ case; the mass and particle number rise with the increasing of $\phi(0)$ to a maximum of $M_{max} = 0.633 M_{Pl}^2/\mu$ and $N_{max} = 0.653 M_{Pl}^2/\mu^2$. After this M and N drop, oscillate a bit and finally approach constant values independent of $\phi(0)$.

From *Figures 1b, 4b* it is clear that the introduction of $\Lambda < 0$ does not change this behaviour qualitatively. The location of maximum shifts to higher values of $\phi(0)$ while the maximum mass and particle number decreases with the value of the cosmological constant. However, the general properties of the solutions are the same as for the $\Lambda = 0$ case. The results for $\Lambda = 0$ agrees with previous results in the literature.

As a general feature, we have noticed a decreasing of the maximal allowed value of the parameters N, M and a smaller ADM mass for the same central density as compared with $\Lambda = 0$ case. For example, when $\Lambda = -0.05$ we have found that $M_{max} = 0.480$ and $N_{max} = 0.435$, while for $\Lambda = -1$, these values are $M_{max} = 0.239$, $N_{max} = 0.121$. Different values for the shooting parameter δ_0 are found.

From *Figure 2b* it is evident that the characteristic boson-star masses decrease with increasing $|\Lambda|$. We can qualitatively understand this fact in the following way. Adding a negative cosmological constant corresponds to adding an attractive force. The Heisenberg uncertainty principles requires that a quantum state confined into a region of characteristic radius R has a typical boson momentum $p \sim 1/R$. Since for a marginally relativistic boson star $p \sim \mu$ we find $R \sim 1/\mu$. The order of magnitude of the critical mass for the formation of a black hole can be estimated by comparing R with the Schwarzschild radius r_s . Since the relation between Schwarzschild black hole mass and event horizon radius is $2M = r_h(1 - \Lambda/3r_h^2)^{-1} \sim r_h(1 + \Lambda/3r_h^2)$, we find a critical mass $M_c \sim (1 + \Lambda/3\mu^2)M_{Pl}^2/\mu$, always smaller than the asymptotically flat value.

Note that, again, for a finite value of the central density, the binding energy of the configurations $E_b \equiv M - N$ becomes positive, signaling the existence of configurations with excess energy. This property express a global instability against dispersion of particles to infinity. The excess energy is translated into kinetic energy of the free particles at infinity.

4.2 Boson stars in five dimensions

We integrated the equations of motion for a number of dimensions greater than four. We find that the general picture we presented for $D = 4$ remains always valid for higher dimensional spherically symmetric bosons stars.

The parameter M rises quickly with the initial value of the scalar field $\phi(0)$ at a maximum, afterwards drops, and has several local minima and maxima until it approaches an asymptotic value for large $\phi(0)$. A similar behaviour is found for the particle number. The location of the first peak again shifts to higher values of $\phi(0)$ for a negative Λ . Curiously, the five-dimensional asymptotically flat solutions present this peak for very small values of φ_0 .

The results for three different values of $\phi(0)$ and a varying Λ are presented in *Figures 2c, 3c*. As expected, the values of (M, N) for a given φ_0 depends on the value of Λ . For a nonzero Λ it is necessary to establish new (and smaller) limiting values for the maximal mass and maximal particle number. For example, an asymptotically flat five-dimensional solution has $M_{max} = 3.803$ and $N_{max} = 3.792$; when $\Lambda = -0.01$ we have found that $M_{max} = 1.764$ and $N_{max} = 1.667$, while for $\Lambda = -0.1$, these values are $M_{max} = 0.893$, $N_{max} = 0.719$. We can heuristically understand this fact by using the arguments presented for $D = 4$. Typical results of the numerical integration are presented in *Figures 1c, 4c*.

Note that again for a finite value of the central density the binding energy of the configurations $E_b \equiv M - N\mu$ becomes positive, signaling the existence of configurations with excess energy.

4.3 Boson stars in three dimensions

The (2+1) dimensional boson stars are a rather special case. Three dimensional gravity provided us with many important clues about higher dimensional physics. It help that this theory with a negative cosmological constant Λ has non-trivial solutions, such as the BTZ black-hole spacetime [43], which provide important testing ground for quantum gravity and AdS/CFT correspondence. Many other types of 3D regular and black hole solutions with a negative cosmological constant have also been found by coupling matter fields to gravity in different way. (2+1)-dimensional stars with negative cosmological constant have been studied in [44, 45]. We notice also the existence of interesting BTZ-like solutions in a class of 3D gravity models in which the cosmological constant is induced [46].

Exact boson star solutions with $\Lambda < 0$ (in the limit of large self interaction) are also known to exist [32].

The line element (8) reads in this case

$$ds^2 = \frac{dr^2}{1 - 2m(r) - \Lambda r^2} + r^2 d\theta^2 - (1 - 2m(r) - \Lambda r^2)e^{-2\delta} dt^2, \quad (33)$$

where the angular variable θ is assumed to vary between 0 and 2π . At the first sight, it may be that the boundary condition (24) is too strong and we have the freedom to add a constant to the function $m(r)$ [32]. However, if we assume that this metric possesses a symmetry axis (located at $r = 0$), and has no conical singularities, we have to impose the condition $\lim_{r \rightarrow 0} m(r) = 0$. Asymptotically, the line element (33) approaches the BTZ metric

$$ds^2 = \frac{dr^2}{-\tilde{M} - \Lambda r^2} + r^2 d\theta^2 - (-\tilde{M} - \Lambda r^2)e^{-2\delta} dt^2, \quad (34)$$

with $\tilde{M} = -1 + 2m(\infty)$. We will find in Section 6 that the interpretation of the function $m(r)$ and of its limit as $r \rightarrow \infty$ is more subtle.

We have integrated the field equations with the initial conditions (23) for a number of $\Lambda < 0$ and have found a different picture as compared to the case $D > 3$. The solution with $\varphi_0 = 0$ corresponds to the global AdS_3 . For a given Λ , we find nontrivial solutions up to a maximal value of φ_0 , where the numerical iteration diverges. A divergent result is obtained also in the limit $\Lambda \rightarrow 0$. The results of the numerical integration both for M and N and three values of Λ are presented in *Figures 1a, 4a*. In *Figures 2a, 3a* we present the variation of M and N with Λ for fixed values of the central density.

We noticed that, for every Λ, ϕ_0 , the asymptotic value of the function $m(r)$ is smaller than $1/2$. However, the maximal allowed value of the particle number can be higher than $1/2$ (for small enough values of $|\Lambda|$).

Further details on these solutions as well as rotating 3D configurations will be presented elsewhere.

5 STABILITY OF SOLUTIONS VIA LINEAR PERTURBATION THEORY

An important physical question when discussing selfgravitating configurations is whether these solutions are stable. For a boson star, the discussion of gravitational equilibrium is more complicated since we cannot apply the well-known stability theorems for fluid stars. When the spacetime is not asymptotically flat the stability analysis can also be a quite involved and subtle problem.

A number of authors discussed the dynamical stability of four dimensional $\Lambda = 0$ boson stars by means of linear perturbation theory [21]-[23]. This lead to an eigenvalue problem, which is of Sturm-Liouville type and which determines the normal modes of the radial oscillations and their eigenvalues χ^2 . The sign of the lowest eigenvalue χ_0^2 is crucial; if $\chi_0^2 > 0$ then the star is stable, whereas it is unstable. In particular Gleiser and Watkins used a powerful method to discuss the stability of asymptotically flat boson stars, by studying the frequency spectrum of a class of radial perturbations [22]. They have shown that the transition from stability to instability occurs always at the critical points of mass (or charge) against the value of the scalar field at the origin. A more powerful method based on catastrophe theory was also proposed [24], with similar conclusions.

Since the general picture we find for $D \geq 4$ is similar to the well known asymptotically flat solutions, we expect to find similar results for the stability question. Therefore we conjecture that M_{max} represents the boundary between stable and unstable gravitational equilibrium.

The method proposed by Gleiser and Watkins [22] will be used here to discuss the stability of D dimensional solutions with $\Lambda \leq 0$. Since our approach is practically similar to that presented in [22], we will present here the main steps only, emphasizing the points where the different asymptotic structure of spacetime is essential. We find that, for any dimension, the perturbation equations reduce to a set of two coupled differential equations.

We consider the situation where the equilibrium configuration is perturbed in a way such that the spherical symmetry is still preserved. These perturbations will give rise to motions in the radial direction.

To conform with previous works in $D = 4$, we use in this section the following parametrization for the spherically symmetric line element

$$ds^2 = e^\lambda dr^2 + r^2 d\Omega_{D-2}^2 - e^\nu dt^2, \quad (35)$$

where ν and λ are functions of r and t only.

The equations governing the small perturbations are obtained by expanding all functions to first order and then by linearizing the equations (5)-(6). We write therefore

$$\begin{aligned} \lambda(r, t) &= \lambda_0(r) + \delta\lambda(r, t), \\ \nu(r, t) &= \nu_0(r) + \delta\nu(r, t), \end{aligned} \quad (36)$$

where $\delta\lambda(r, t)$, $\delta\nu(r, t)$ are related to the perturbations of m , δ . The scalar field is written as

$$\phi(r, t) = (\phi_0(r) + \delta\phi(r, t))e^{-i\omega t}. \quad (37)$$

Similar to the $\Lambda = 0$ four dimensional case, it is possible to reduce the full system of equations to only two coupled equations in two unknowns. If we choose to obtain these equations for $\delta\phi$, it is convenient to introduce the notations $\delta\phi(r, t) = f_1(r, t) + i\phi_0(r)\dot{g}(r, t)$ (in some cases, the analytically equivalent set of equations for $\delta\lambda$, f_1 may be more suited in the numerical computation). For the equilibrium configurations the metric functions are time independent, $e^{\lambda_0} = 1/F$, $e^{\nu_0} = e^{-2\delta}F$ and the scalar field is $\phi(r, t) = \phi_0(r)e^{-i\omega t}$.

For this scalar field ansatz, the linearized (r, t) Einstein equation can be trivially integrated in time giving

$$\delta\lambda = \frac{2r}{D-2} 16\pi G_D (\phi_0' f_1 - \omega \phi_0^2 g'). \quad (38)$$

The (r, r) and (t, t) Einstein equations give two more equations for the metric perturbations

$$-\frac{(D-2)}{2r^2} [(D-3)e^{-\lambda_0} \delta\lambda + re^{-\lambda_0} \delta\lambda' - re^{-\lambda_0} \lambda_0' \delta\lambda] = 8\pi G_D [e^{-\nu_0} \delta\nu \omega^2 \phi_0^2 \quad (39)$$

$$\begin{aligned}
& +e^{-\lambda_0}\delta\lambda\phi_{0,r}^2 - 2e^{-\nu_0}\omega\phi_0(\omega f_1 - \phi_0\ddot{g}) - 2e^{-\lambda_0}\phi_0'f_1' - 2\mu^2\phi_0f_1 \Big], \\
\delta\nu' &= (D-3)\frac{\delta\lambda}{r} + \nu_0'\delta\lambda + \frac{2r}{D-2}8\pi G_D e^{\lambda_0} \Big[-e^{-\lambda_0}\phi_0'^2\delta\lambda + 2e^{-\lambda_0}\phi_0'f_1' - 2\mu^2\phi_0f_1, \\
& -e^{-\nu_0}\omega^2\phi_0^2\delta\nu + 2e^{-\nu_0}\omega\phi_0(\omega f_1 - \phi_0\ddot{g}) \Big],
\end{aligned} \tag{40}$$

while the linearized Klein-Gordon equation for f_1 and g are

$$f_1'' + f_1' \left(\frac{D-2}{r} + \frac{1}{2}(\nu_0' - \lambda_0') \right) - e^{\lambda_0-\nu_0}\ddot{f}_1 + f_1 e^{\lambda_0}(\omega^2 e^{-\nu_0} - \mu^2) - 2\omega e^{\lambda_0-\nu_0}\phi_0\ddot{g} \tag{41}$$

$$+ \frac{1}{2}(\delta\nu' - \delta\lambda') + e^{\lambda_0-\nu_0}\omega^2\phi_0(\delta\lambda - \delta\nu) - \mu^2 e^{\lambda_0}\phi_0\delta\lambda = 0,$$

$$g'' + g' \left[\frac{D-2}{r} + \frac{1}{2}(\nu_0' - \lambda_0') + 2\frac{\phi_0'}{\phi_0} \right] - \ddot{g} e^{\lambda_0-\nu_0} + 2\omega e^{\lambda_0-\nu_0}\frac{f_1}{\phi_0} + \frac{1}{2}\omega e^{\lambda_0-\nu_0}(\delta\lambda - \delta\nu) = 0. \tag{42}$$

The procedure now is simple. Using eq. (38), it is possible to eliminate $\delta\lambda$ from (41). An expression for $\delta\nu$ is obtained for eq. (42). We can also differentiate eq. (42) with respect to r and eliminate $\delta\nu'$ by using eq. (40). In this way we find

$$f_1'' + f_1' \left(\frac{D-2}{r} + \frac{1}{2}(\nu_0' - \lambda_0') \right) - \ddot{f}_1 e^{\lambda_0-\nu_0} + C_1(r)f_1 - 2\omega\phi_0g'' - C_3(r)g' \tag{43}$$

$$- \omega g' \left(2\phi_0' + \phi_0 \left(\frac{2D-4}{r} + \nu_0' - \lambda_0' \right) \right) = 0,$$

$$g''' + g'' \left[\frac{D-2}{r} + \frac{3}{2}(\nu_0' - \lambda_0') + \frac{\phi_0'}{\phi_0} \right] - e^{\lambda_0-\nu_0}\ddot{g}' + 2e^{\lambda_0-\nu_0}\frac{\omega}{\phi_0}f_1' - C_2(r)g' - \frac{e^{\lambda_0-\nu_0}}{\phi_0^2}C_3(r)f_1 = 0, \tag{44}$$

where

$$\begin{aligned}
C_1 &= -e^{\lambda_0}(\mu^2 + 3\omega^2 e^{-\nu_0}) + \frac{16\pi G_D}{D-2}r\phi_0'^2 \left(\frac{2D-6}{r} + \nu_0' - \lambda_0' \right) - \frac{64\pi G_D}{D-2}re^{\lambda_0}\phi_0'\phi_0\mu^2, \\
C_2 &= (D-2)\frac{(\lambda_0' - \nu_0')}{r} + \frac{D-2}{r^2} - \frac{1}{2}(\nu_0'' - \lambda_0'') + 2\left(\frac{\phi_0'}{\phi_0}\right)^2 - 2\frac{\phi_0''}{\phi_0} \\
&+ \frac{1}{2}(\lambda_0' - \nu_0')(\nu_0' - \lambda_0' + 4\frac{\phi_0'}{\phi_0}) - \frac{16\pi G_D}{D-2}re^{\lambda_0-\nu_0}\omega^2\phi_0^2(\nu_0' - \lambda_0' + \frac{2D-6}{r}), \\
C_3 &= \omega \left(2\phi_0' + \frac{16\pi G_D}{D-2}r\phi_0^2\phi_0' \left(\frac{2D-6}{r} + \nu_0' - \lambda_0' \right) - \frac{32\pi G_D}{D-2}r\phi_0^3\mu^2 e^{\lambda_0} \right).
\end{aligned} \tag{45}$$

We consider only radial perturbations, which conserve the total particle number N , as given by (20), from which we can compute δN . Remarkably, we find that always the integrant of δN is a total derivative with respect to r

$$\delta N = \frac{8\pi^{(D-1)/2}}{\Gamma(\frac{D-1}{2})}\phi_0^2 g' r^{D-2} e^{\frac{\nu_0-\lambda_0}{2}} \Big|_0^\infty. \tag{46}$$

The appropriate boundary conditions for $r \rightarrow \infty$ are $f_1 \rightarrow 0$, $g' r^{D+2c} \rightarrow 0$ while $f_1 = \text{const.}$, $r^2 g' \rightarrow 0$ at the origin.

Now we assume that all perturbations are of the form $e^{i\chi t}$, where χ is the characteristic frequency to be determined. The self-adjoint system of coupled equation (43) and (44) can be written in the compact form

$$L_{ij}f_j = \chi^2 M_{ij}f_j, \quad i, j = 1, 2 \tag{47}$$

with $f_2 = g'$ and

$$L_{ij} = \begin{pmatrix} -\frac{d}{dr}G_1\frac{d}{dr} + G_1C_1 & 2\omega\frac{d}{dr}G_1\phi_0 + G_1C_3 \\ -2\omega G_1\phi_0\frac{d}{dr} + G_1C_3 & -\frac{d}{dr}G_2\frac{d}{dr} + G_2C_2 \end{pmatrix}, \quad M_{ij} = e^{\lambda_0-\nu_0} \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix},$$

where $G_1 = r^{D-2}e^{(\nu_0-\lambda_0)/2}$, $G_2 = r^{D-2}\phi_0^2e^{3(\nu_0-\lambda_0)/2}$, which is the required "pulsation equation". This system defines a characteristic value problem for χ^2 which must be real (note the close analogy with the $D = 4, \Lambda = 0$ case). With the above boundary conditions, it can be easily verified that both L_{ij} and M_{ij} satisfy the condition

$$\int_0^\infty dr \eta_i L_{ij} \xi_j = \int_0^\infty dr \xi_i L_{ij} \eta_j,$$

and similarly for M_{ij} . The equations (43) and (44) can also be derived from a variational principle. Eigenfunctions related to different eigenvalues satisfy also an orthogonality relation. The eigenvalues χ^2 form a discrete sequence $\chi_0^2 \leq \chi_1^2 \leq \chi_2^2 \dots$. Thus, if the fundamental mode is stable ($\chi_0^2 \geq 0$), then all radial modes are stable. Also, if the boson star is radially unstable, the fastest growing instability will be the fundamental one. We are interested in finding the values of $\phi_0(0)$ for which the equilibrium configurations are stable. As extensively discussed in [22], we can find some insight in this problem by considering static perturbation. If for a given equilibrium configuration a radial mode changes its stability property (therefore its corresponding eigenvalues χ_0^2 passes through zero), then there will exist infinitesimally nearby equilibrium configurations with the same total mass and total particle number. Hence, if χ^2 goes through zero we have $dM/d\phi_0(0) = dN/d\phi_0(0) = 0$.

Also, if we begin with an equilibrium configuration with ϕ_0 , the perturbed fields will describe another equilibrium configuration with $\phi_0 + \delta\phi_0$. However, the perturbed configurations must have the same charge as the equilibrium configurations. Therefore, zero frequency perturbations will exist if and only if there exist two neighboring configurations with the same charge. This occurs at the extremum of N , where $dN/d\phi_0(0) = 0$. This is a very general argument and applies for every dimension.

For all discussed cases except $D = 3$, the behaviour of the mass and particle number, parametrized by φ_0 suggest that the solutions with φ_0 smaller than $\phi_c(0)$ corresponding to the maximum mass, are stable, whereas the ones with bigger values than $\phi_c(0)$ are unstable. Thus we expect that the value of φ_0 corresponding to the maximal mass is the boundary between stable and unstable equilibrium configurations.

Of course, a complete proof still requires an analysis of the eigenvalues of the pulsation equations for given (D, Λ) in order to establish if they are real and positive for central densities smaller than $\phi_c(0)$. However, it is a tedious procedure to find the characteristic values of χ^2 for a given value of ϕ_0 (see [47] for a detailed discussion of this problem in $D = 4, \Lambda = 0$ case). For our case, we solved the equations (43), (44) in a neighbourhood of $\phi_c(0)$, for $D = 4$ and several negative values of Λ (we consider nodeless unperturbed solutions only). The boundary conditions are as follows: for $r \rightarrow 0$ we find

$$\begin{aligned} f_1 &= p_0 + p_2 r^2 + O(r^3), \quad g' = q_1 r + O(r^2), \\ p_2 &= \omega \varphi_0 q_1 + \frac{p_0}{2(D-1)} \left(-\chi^2 e^{-\nu_0(0)} + \mu^2 + 3e^{-\nu_0(0)} \omega^2 \right). \end{aligned} \quad (48)$$

We can use the linearity of the equations (43), (44) to set $f_1(0) = 1$. Dimensionless quantities are obtained by using the same rescaling discussed in Section 2. We then have two parameters left χ^2 and q_1 and the solution of the problem can be found by using a standard shooting procedure. The numerical results in this case confirm that $\chi^2 > 0$ for central values of the scalar field smaller than $\phi_c(0)$ while other configurations are unstable ($\chi^2 < 0$). In Figure 5 we present the result of the numerical integration for the perturbation of the metric function λ for three different configurations with maximal mass ($\chi^2 = 0$). The variation of χ^2 with the central value of the scalar field is presented in Figure 6 for two negative values of the cosmological constant. We have all reasons to expect to find a similar result for any $\Lambda \leq 0$ and $D > 3$.

6 THE MASS AND THE BOUNDARY STRESS TENSOR

An important problem of AdS space concerns the definition of mass and angular momentum of AAdS spacetimes. The generalization of Komar's formula in this case is not straightforward and requires the further subtraction of a background configuration in order to render a finite result. This problem was addressed for the first time in the eighties, with different approaches (see for instance Ref. [48, 49]).

At spatial infinity, the line element (8) can be written as

$$ds^2 = ds_0^2 + h_{\mu\nu} dx^{\mu\nu}, \quad (49)$$

where $h_{\mu\nu}$ are deviations from the background AdS metric ds_0^2 . Similar to the asymptotically flat case, one expects the values of mass and other conserved quantities to be encoded in the functions $h_{\mu\nu}$.

Using the Hamiltonian formalism, Henneaux and Teitelboim [49] have computed the mass of an AAdS spacetime in the following way. They showed that the Hamiltonian must be supplemented by surface terms in order to be consistent with the equations of motion. These surface terms yield conserved charges associated with the Killing vectors of an AAdS geometry. The total energy is the charge associated with the Killing vector $\partial/\partial t$. A similar result has been obtained by Abbott and Deser [48], in a more general context (see Ref. [50] for a discussion of the relation between the Hamiltonian mass and the Abbott-Deser one). This method requires to choose a reference background with suitable matching conditions. However, in some interesting cases (such as configurations with NUT charge) this choice is ambiguous.

Another formalism to define conserved charges in AAdS spacetimes was proposed in [51] and uses conformal techniques to construct conserved quantities. This construction makes no reference to an action. However, it yields the results obtained by Hamiltonian methods.

A different approach has been proposed recently for locally AdS_4 asymptotically geometries and generalized in [53] for asymptotically AdS_{2n} spacetimes (see also [54]). It consists in adding an Euler (boundary-)term to the gravitational action [52]. The purpose is to guarantee that the action reaches an extremum for solutions which are locally AdS at the boundary. The resulting action gives the mass and angular momentum as Noether charges associated to the asymptotic Killing vectors $\partial/\partial t$ and $\partial/\partial\varphi$. In this formalism the specification of a reference background is also unnecessary. The efficiency of this approach has been demonstrated in a broad range of examples, including Kerr-AdS and Reissner-Nordström-AdS black holes, black strings and Taub-NUT and Taub-Bolt solutions.

As expected, these different methods yield the same total mass for the spherically symmetric AAdS boson star solutions considered in this paper (for computations using Hamiltonian methods we consider the obvious vacuum AdS_D background)

$$M_{ADM} = \frac{(D-2)\Omega_{D-2}}{8\pi G_D} M, \quad (50)$$

where $\Omega_{D-2} = 2\pi^{(D-1)/2}/\Gamma((D-1)/2)$ is the area of a unit $(D-2)$ -dimensional sphere.

A procedure leading (for odd dimensions) to a different result has been proposed by Balasubramanian and Kraus [34]. This technique was inspired by AdS/CFT correspondence and consists also in adding suitable counterterms I_{ct} to the action in order to ensure the finiteness of the stress tensor derived by the quasilocal energy definition [55]. These counterterms are built up with curvature invariants of a boundary $\partial\mathcal{M}$ (which is sent to infinity after the integration) and thus obviously they do not alter the bulk equations of motion.

Given the potential relevance of these boson star solutions in an AdS/CFT context, we present here a computation of the boundary stress tensor and of the total mass, by using the counterterm prescription.

The following counterterms are sufficient to cancel divergences in a pure gravity theory for $D \leq 6$, with several exceptions (see e. g. [56]):

$$I_{ct} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{D-1}x \sqrt{-h} \left[\frac{D-2}{l} + \frac{l}{2(D-3)} \mathcal{R} \right]. \quad (51)$$

Here \mathcal{R} is the Ricci scalar for the boundary metric h . Our conventions with respect to indices will be: $\{A, B, \dots\}$ indicate the intrinsic coordinates of the boundary and $\{a, b, \dots\}$ indicate the angular coordinates. In this section we will define also $l^2 = -(D-2)(D-1)/(2\Lambda)$.

Using these counterterms one can construct a divergence-free stress tensor from the total action $I = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}}$, by defining (see e.g. Ref. [57])

$$T_{AB} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{AB}} = \frac{1}{8\pi G} (K_{AB} - K h_{AB} - \frac{D-2}{l} h_{AB} + \frac{l}{D-3} E_{AB}), \quad (52)$$

where E_{AB} is the Einstein tensor of the intrinsic metric h_{AB} .

We can use this result to assign a mass to an AAdS geometry by writing the boundary metric in an ADM form

$$h_{AB}dx^A dx^B = -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\sigma^a dt)(dx^b + N_\sigma^b dt) \quad (53)$$

and the definition

$$\bar{M} = \int_{\partial\Sigma} d^{D-1}x \sqrt{\sigma} N_\Sigma \epsilon. \quad (54)$$

Here $\epsilon = u^\mu u^\nu T_{\mu\nu}$ is the proper energy density while u^μ is a timelike unit normal to Σ .

The metric restricted to the boundary h_{AB} diverges due to an infinite conformal factor r^2/l^2 . The background metric upon which the dual field theory resides is

$$\gamma_{AB} = \lim_{r \rightarrow \infty} \frac{l^2}{r^2} h_{AB}. \quad (55)$$

For the asymptotically AdS_D solutions considered here, the $(D-1)$ dimensional boundary is the Einstein universe, with the line element

$$\gamma_{AB}dx^A dx^B = -dt^2 + l^2 d\Omega_{D-2}^2. \quad (56)$$

If there are matter fields on \mathcal{M} , additional counterterms may be needed to regulate the action. However, we find that for a boson star spacetime in $D = 3, 4, 5$ dimensions, the prescription (51) removes all divergences (this is a consequence of the asymptotic behavior of the scalar field (26)). The results we find for the boundary stress tensor of a boson star at large r by using the asymptotic expressions (26) are

$$T_{ab}^{(3)} = l(M - \frac{1}{2})\delta_{ab} + \dots, \quad (57)$$

$$T_{tt}^{(3)} = \frac{1}{l}(M - \frac{1}{2}) + \dots, \quad (58)$$

in three dimensions and

$$T_{ab}^{(4)} = \frac{lM}{r}\omega_{ab} + \dots \quad (59)$$

$$T_{tt}^{(4)} = \frac{2M}{lr} + \dots, \quad (60)$$

for $D = 4$. The boundary stress tensor for the five-dimensional case is

$$T_{ab}^{(5)} = (M + \frac{l^2}{8})\frac{l}{r^2}\omega_{ab} + \dots \quad (61)$$

$$T_{tt}^{(5)} = \frac{3M}{lr^2} + \frac{3l}{8r^2} + \dots, \quad (62)$$

By using the relation (54), we can find the mass of boson star solutions, according to Balasubramanian and Kraus. The mass of a four-dimensional boson star is $\bar{M} = M$, predicted also by (50). The result for $D = 3$ is $\bar{M} = M - 1/2$, while in five dimensions we find $\bar{M} = M_{ADM} + 3\pi l^2/32$. However, in $D = 5$, the standard value is M_{ADM} . The additional constant $3\pi l^2/32$ is the mass of pure global AdS_5 and is usually interpreted as the energy dual to the Casimir energy of the CFT defined on a four dimensional Einstein universe [34].

For $D = 3$, we find that the mass of solutions computed in this way, is always larger than -1 , with the extreme value $\bar{M} = -1$ corresponding to the global AdS_3 space. Contrary to what happens for BTZ black holes, we do not find a mass gap between $\bar{M} = -1$ and $\bar{M} = 0$. To compute the mass for this type of configuration, usually one uses a matching procedure on a surface separating the regions where the internal and external geometries are defined (see *e.g.* [32]). The external geometry is taken to be the BTZ black

hole which gives a value for the mass of the solutions. The counterterm method gives, however, a rigorous prescription to compute the total mass of a 3D star, without the rather unnatural matching procedure.

In light of the AdS/CFT correspondence, Balasubramanian and Kraus have interpreted Eq. (52) as $\langle \tau^{AB} \rangle = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{eff}}{\delta \gamma_{AB}}$, where $\langle \tau^{AB} \rangle$ is the expectation value of the CFT stress tensor. Then, the divergences which appear are simply the standard ultraviolet divergences of a quantum field theory and we can cancel them by adding local counterterms to the action. Corresponding to the boundary metric (56), the stress-energy tensor τ_{AB} for the dual theory can be calculated using the following relation [58]

$$\sqrt{-\gamma} \gamma^{AB} \langle \tau_{BC} \rangle = \lim_{r \rightarrow \infty} \sqrt{-h} h^{AB} T_{BC}. \quad (63)$$

However, further progress in this direction seems difficult, since we do not know the underlying boundary CFT for the action (1). In particular, we do not know what the gravitating complex scalar field corresponds to in CFT language.

7 CONCLUDING REMARKS

Many particle theories predict that weakly interacting bosonic particles are abundant in the universe, and that dark matter can be bosonic in nature.

In this paper we investigated the basic properties of boson star solutions in the presence of a negative cosmological constant. Both analytical and numerical arguments have been presented for the existence of nontrivial solutions. A general virial relation has been found and particular cases have been discussed.

The results we find for $D \geq 4$ are broadly similar to those valid for the known asymptotically flat, four dimensional case and there are only some small differences. Nontrivial solutions are again found for every value of the central scalar field and discrete values of frequencies while the scalar field vanishes at infinity. However, a nonzero Λ term in the action implies a complicated power decay of the fields at infinity, rather than exponentially as expected. Also the parameters of the solutions in the asymptotically flat case (mass, particle number, effective radius) are no longer valid and take new values for every choice of Λ . The mass of the star decreases with the cosmological constant strength $|\Lambda|$. Therefore the existence of a negative cosmological constant implies a decrease in the maximal mass of the star. In the sense of the no-scalar-hair theorem we have shown that, for this system, the only spherically symmetric black hole solution in D -dimensions is the Schwarzschild-AdS solution.

These results are not so surprising, since the different asymptotic structure of spacetime obtained for a $\Lambda \neq 0$ does not manifest here directly. The existence of the term $V(\Phi)\sqrt{-g}$ in the action (1) still implies $\Phi \rightarrow 0$ asymptotically (since $g_{rr}g_{tt} \rightarrow -1$), independently on the presence of Λ .

A similar mechanism will act also for a Higgs scalar interacting with a nonabelian field in an AAdS geometry [6, 8]. Here we consider only the better known case of a $SU(2)$ field in four dimensions. In this case, a spherically symmetric purely magnetic YM field can be written as (see e.g. [4])

$$A = \frac{1}{2} \left[\omega(r) \tau_1 d\theta + (\cot \theta \tau_3 + \omega(r) \tau_2) \sin \theta d\varphi \right] \quad (64)$$

(where ω is the YM potential and τ_i are the usual Pauli matrices), which implies an YM effective lagrangian $L_{YM} = 1/2 e^{-\delta} (F\omega'^2 + V(\omega))$, with $V(\omega) = (\omega^2 - 1)^2/2r^2$. The scalar field effective lagrangian is $L_\phi = r^2 e^{-\delta} (F\phi'^2 + V(\phi))$, with $V(\phi)$ the standard Higgs potential. The lagrangian of the system contains also an interacting term $L_i = e^{-\delta} T(\omega) \phi^2$, where $T(\omega) = (\omega + 1)^2$ for a doublet Higgs field [8] and $T(\omega) = \omega^2$ for a Higgs field in the adjoint representation [6]. Again the finite energy requirement forces $V(\phi) \rightarrow 0$ as $r \rightarrow \infty$, which implies that the Higgs field approaches asymptotically the vacuum expectation value. Moreover, the interacting term $T(\omega) \phi^2$ implies that the allowed values of the YM potential at infinity are similar to the $\Lambda = 0$ case. In particular, the solutions are found again for a discrete set of the parameters that specify the initial conditions at the origin. Without a scalar field, the YM behavior is very different. The resulting YM equation reads $(e^{-\delta} (F\omega'))' = e^{-\delta} \omega(\omega^2 - 1)/r^2$ and the YM potential ω takes arbitrary values at infinity $\omega = \omega_0 + w_1/r + O(1/r^2)$. The solutions are found for continuous sets of shooting parameters and have a nonzero magnetic charge [4].

Thus, it seems that when studying a gravitating scalar system, the solutions exhibit a generic behavior for $\Lambda \leq 0$. Given the presence of a cosmological event horizon, the case $\Lambda > 0$ needs a separate analysis.

We noticed, however, that there is a curious change of behaviour for $D = 3$, with somewhat different solution properties.

We expect also to obtain a very similar qualitative behavior of the solutions when discussing a number of possible extensions of this theory, *e.g.* including a dilaton or a selfinteraction term.

Acknowledgement

We would like to thank J.J. van der Bij for valuable discussions and Robert C. Myers for his critical reading of our paper. We thank Dave Winters for proof-reading an earlier draft of this paper.

The work of one of the authors (E.R.) was performed in the context of the Graduiertenkolleg of the Deutsche Forschungsgemeinschaft (DFG): Nichtlineare Differentialgleichungen: Modellierung, Theorie, Numerik, Visualisierung. D.A. would like to thank the Perimeter Institute for Theoretical Physics and the University of Waterloo's Department of Physics for their kind hospitality during this work.

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Figure Captions

Figure 1:

The mass-parameter M (dotted line) and the particle number N (solid line) are represented as a function of $\phi(0)$ for typical boson stars with $\Lambda \leq 0$ in three- (figure 1a), four (figure 1b) and five dimensions (figure 1c). In figures 1-4 the mass parameter M is given in units $\frac{(D-2)\pi^{(D-3)/2}}{4\Gamma((D-1)/2)} \frac{M_{Pl}^{D-2}}{\mu^{D-3}}$, while the particle number N is given in units $\frac{(D-2)\pi^{(D-3)/2}}{4\Gamma((D-1)/2)} (\frac{M_{Pl}}{\mu})^{D-2}$.

Figure 2:

The mass-parameter M is represented as a function of the negative cosmological constant for different values of the central scalar field φ_0 for boson star solutions in three- (figure 2a), four (figure 2b) and five dimensions (figure 2c).

Figure 3:

The particle number N is represented as a function of the negative cosmological constant for different values of the central scalar field φ_0 for boson star solutions in three- (figure 3a), four (figure 3b) and five dimensions (figure 3c).

Figure 4:

The particle number N is represented as a function of the effective radius R (in units $1/\mu$) for different values of the cosmological constant in three- (figure 4a), four (figure 4b) and five dimensions (figure 4c).

Figure 5:

The perturbation of the metric function $\delta\lambda$ is shown as a function of the radial coordinate for the fundamental mode and several values of Λ .

Figure 6:

The squared frequency χ^2 of the perturbations is plotted as a function of scalar field at the origin for $D = 4$ asymptotically AdS boson star solutions with $\Lambda = -0.05, -0.5$. Note that χ^2 is zero when $\Phi(0) \simeq 0.48$ for $\Lambda = -0.05$ and $\Phi(0) \simeq 0.652$ for $\Lambda = -0.5$, which corresponds to boson stars with maximum possible mass.

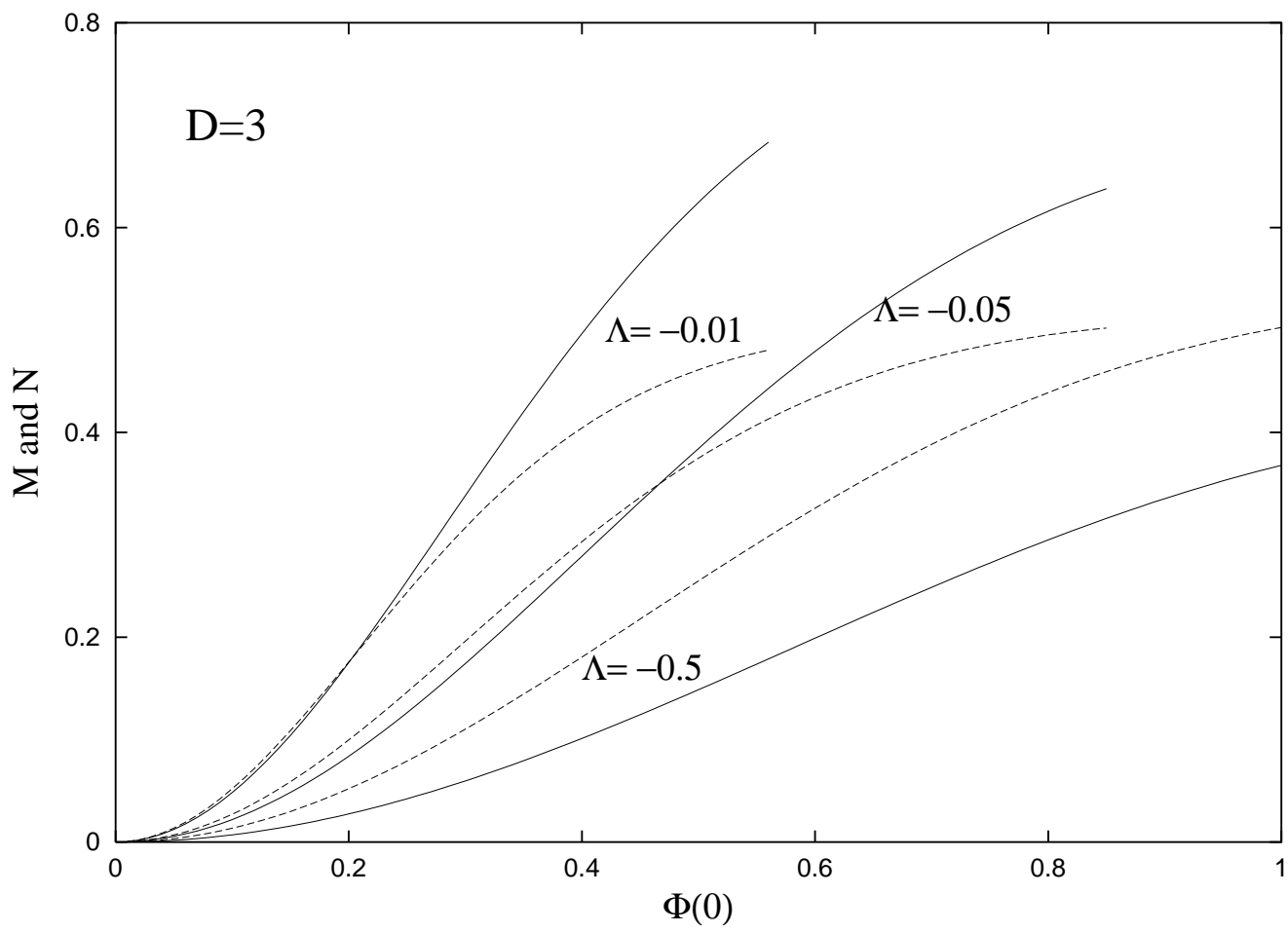


Figure 1a.

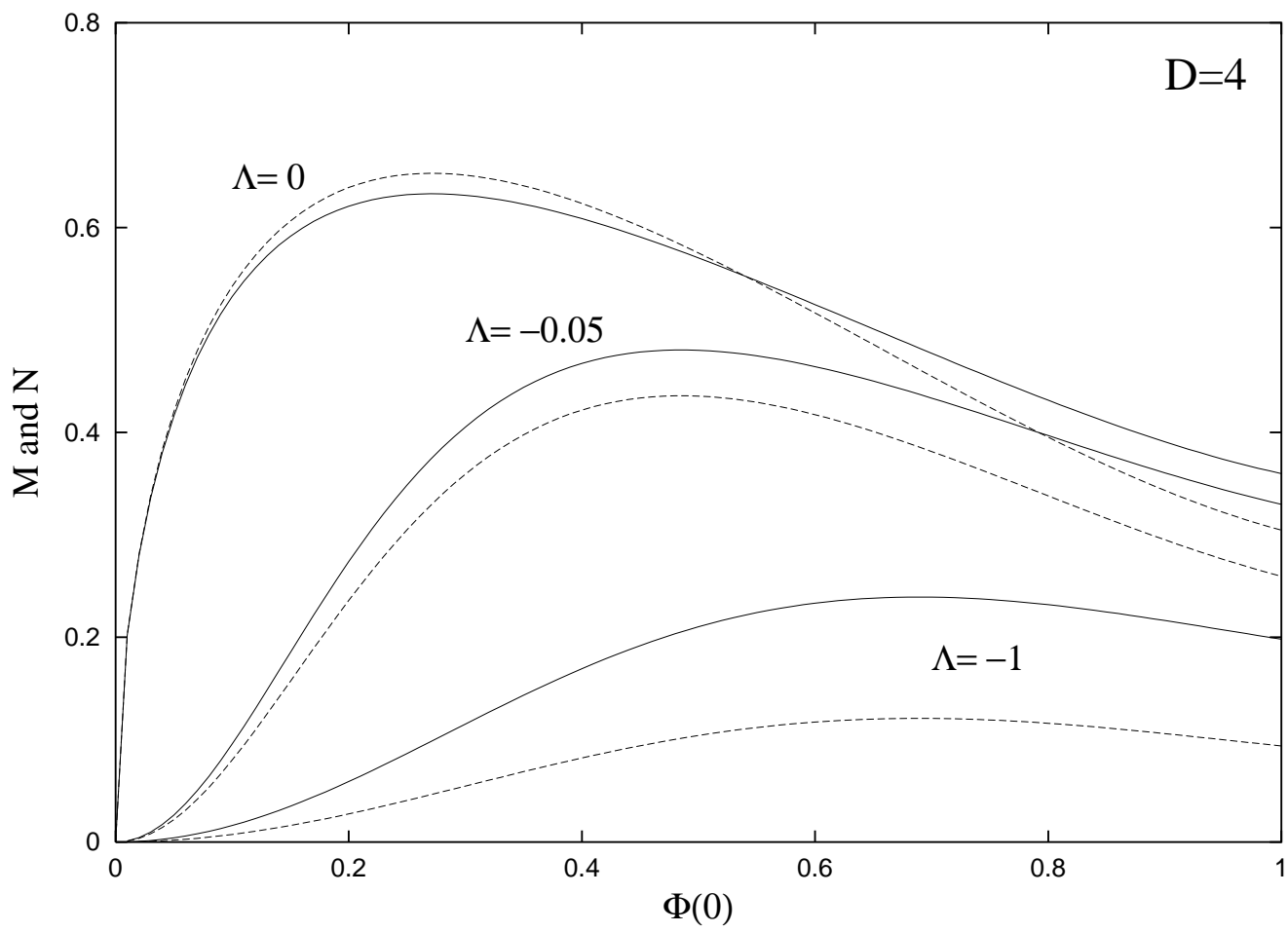


Figure 1b.

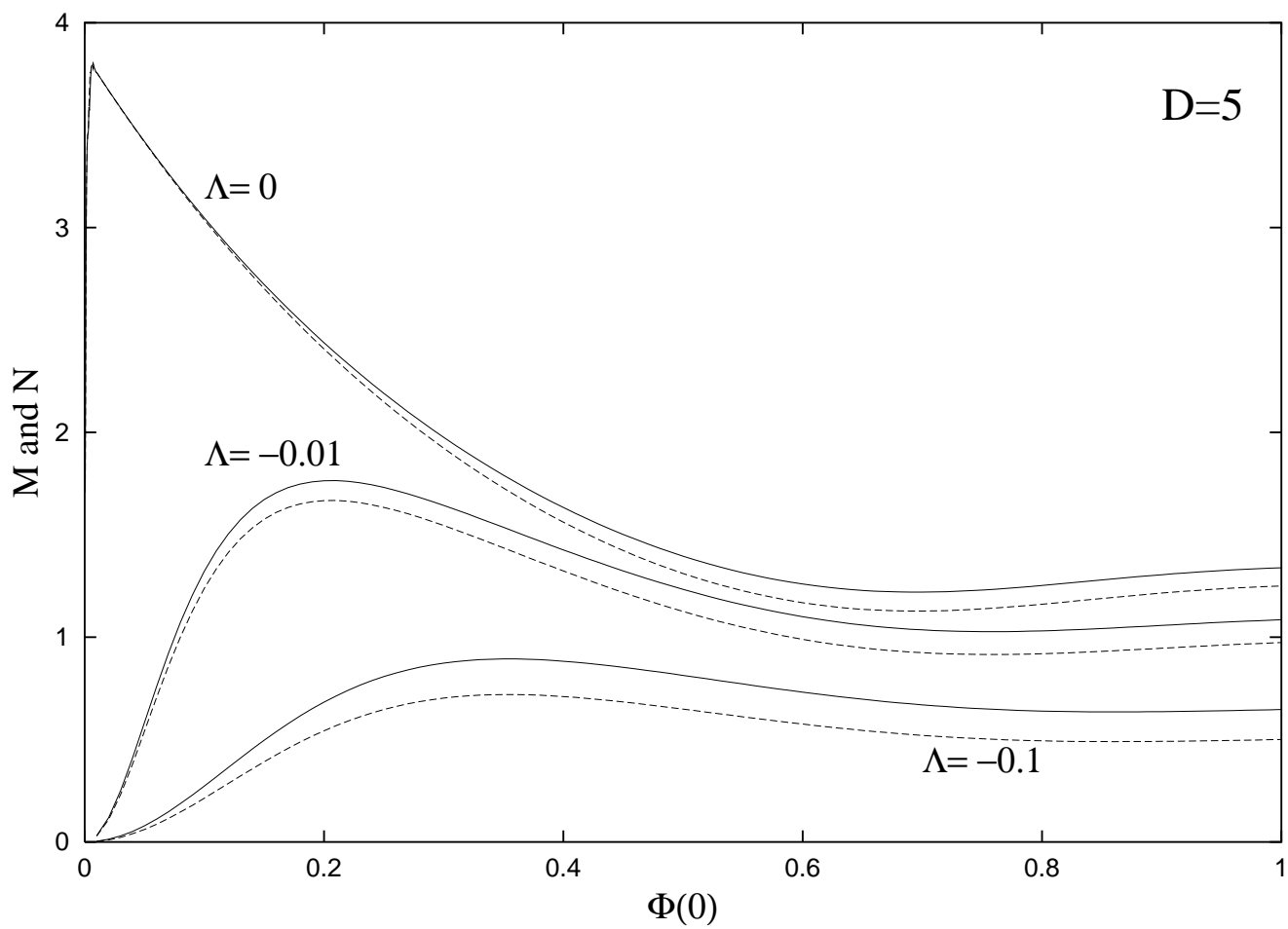


Figure 1c.

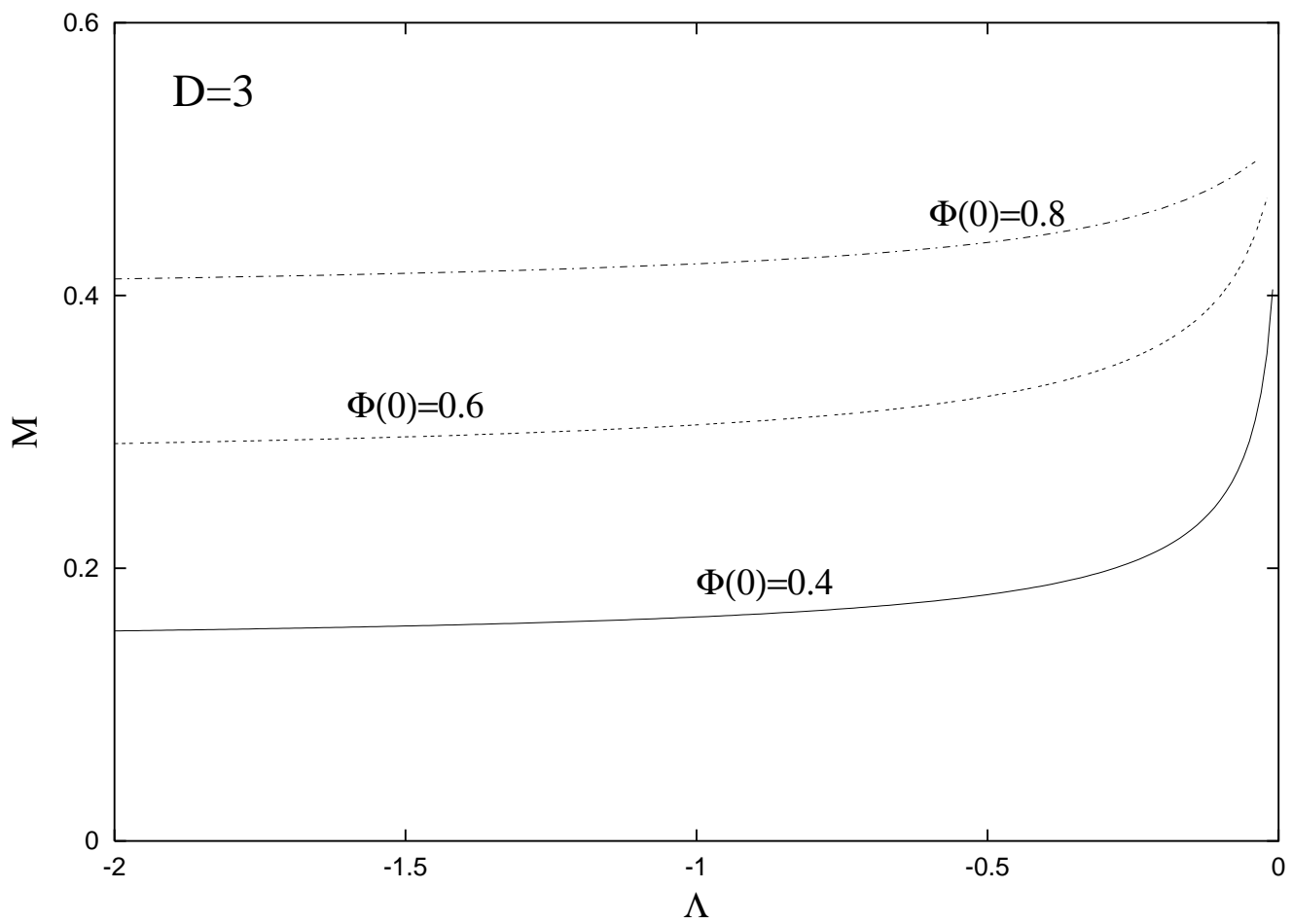


Figure 2a.

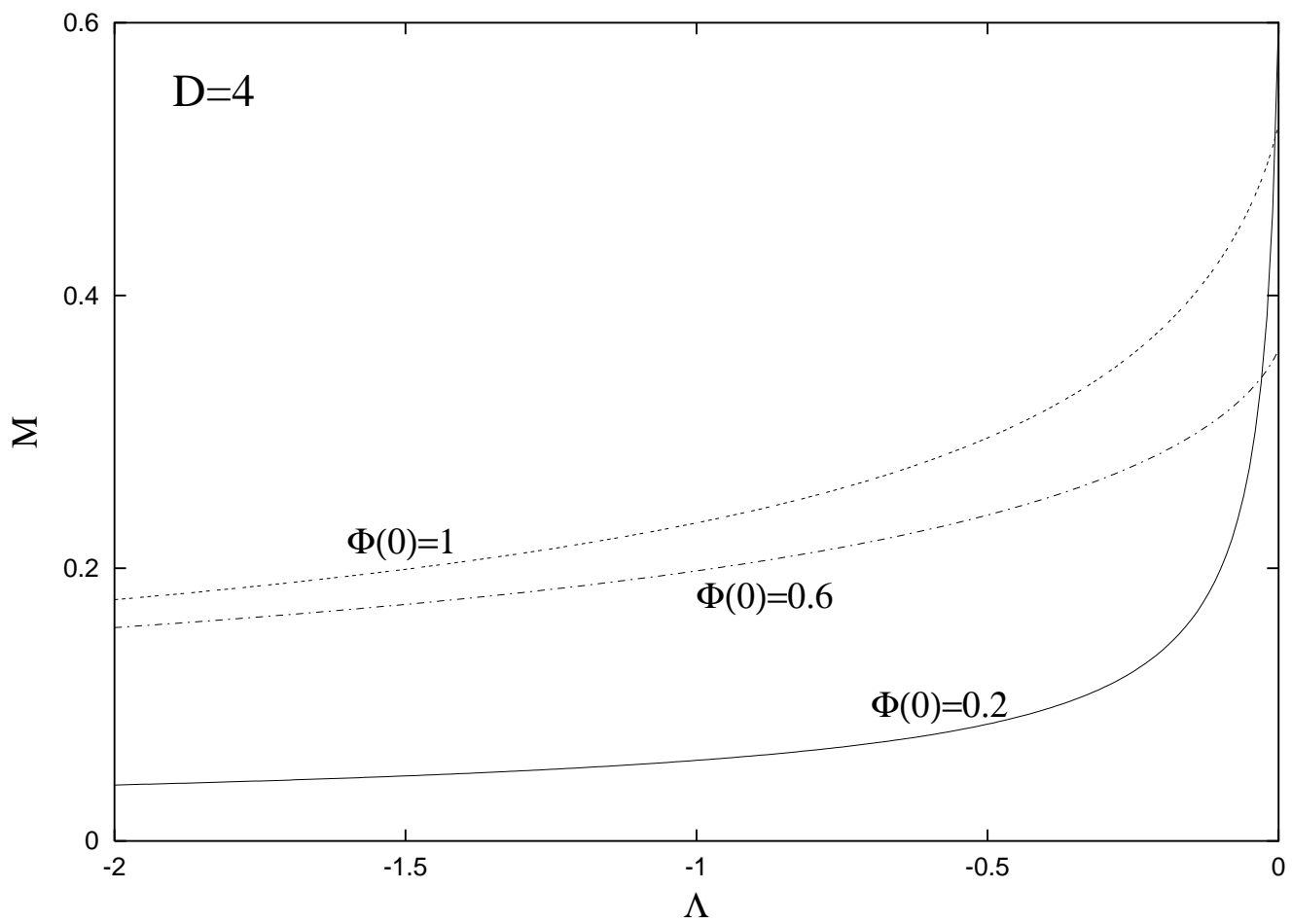


Figure 2b.

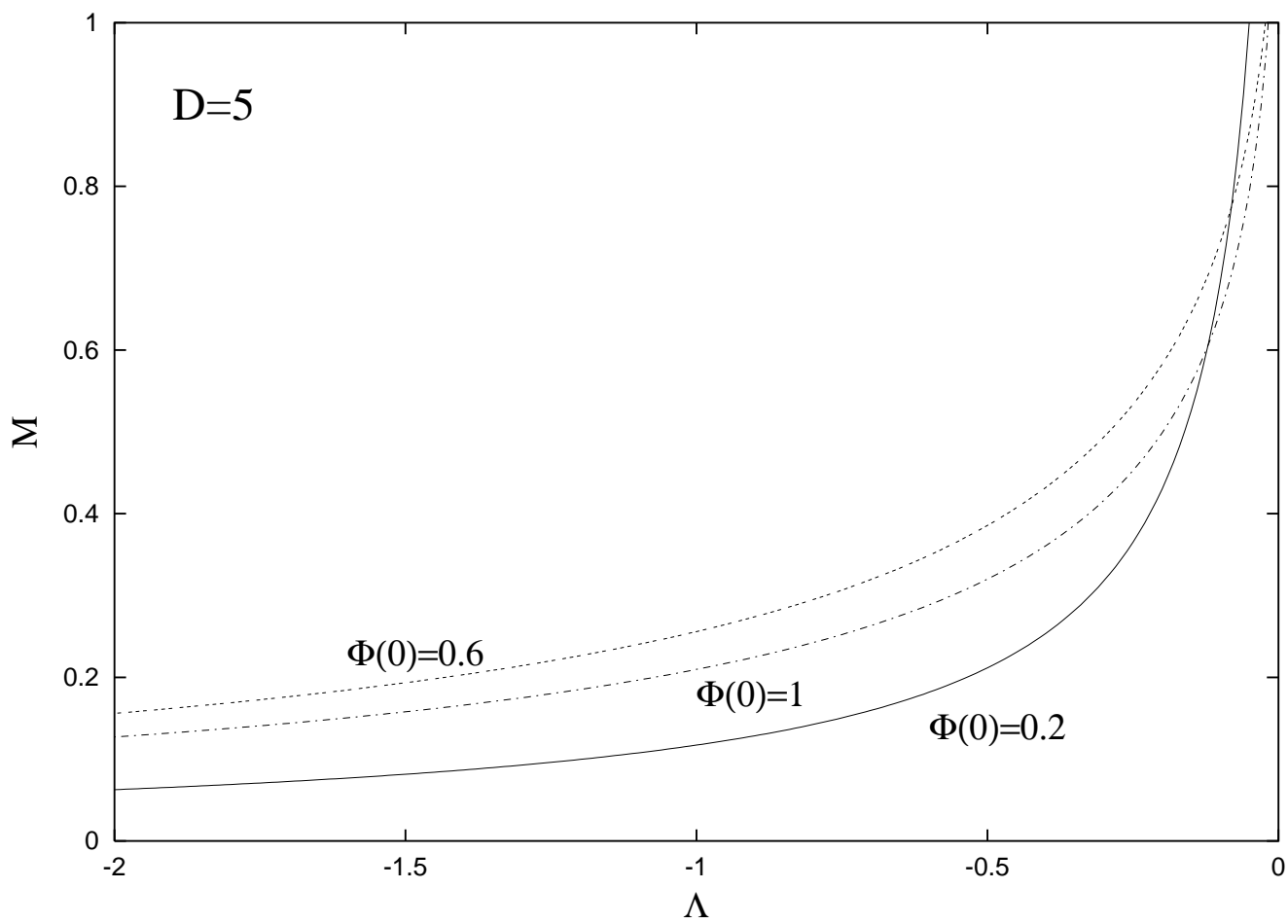


Figure 2c.

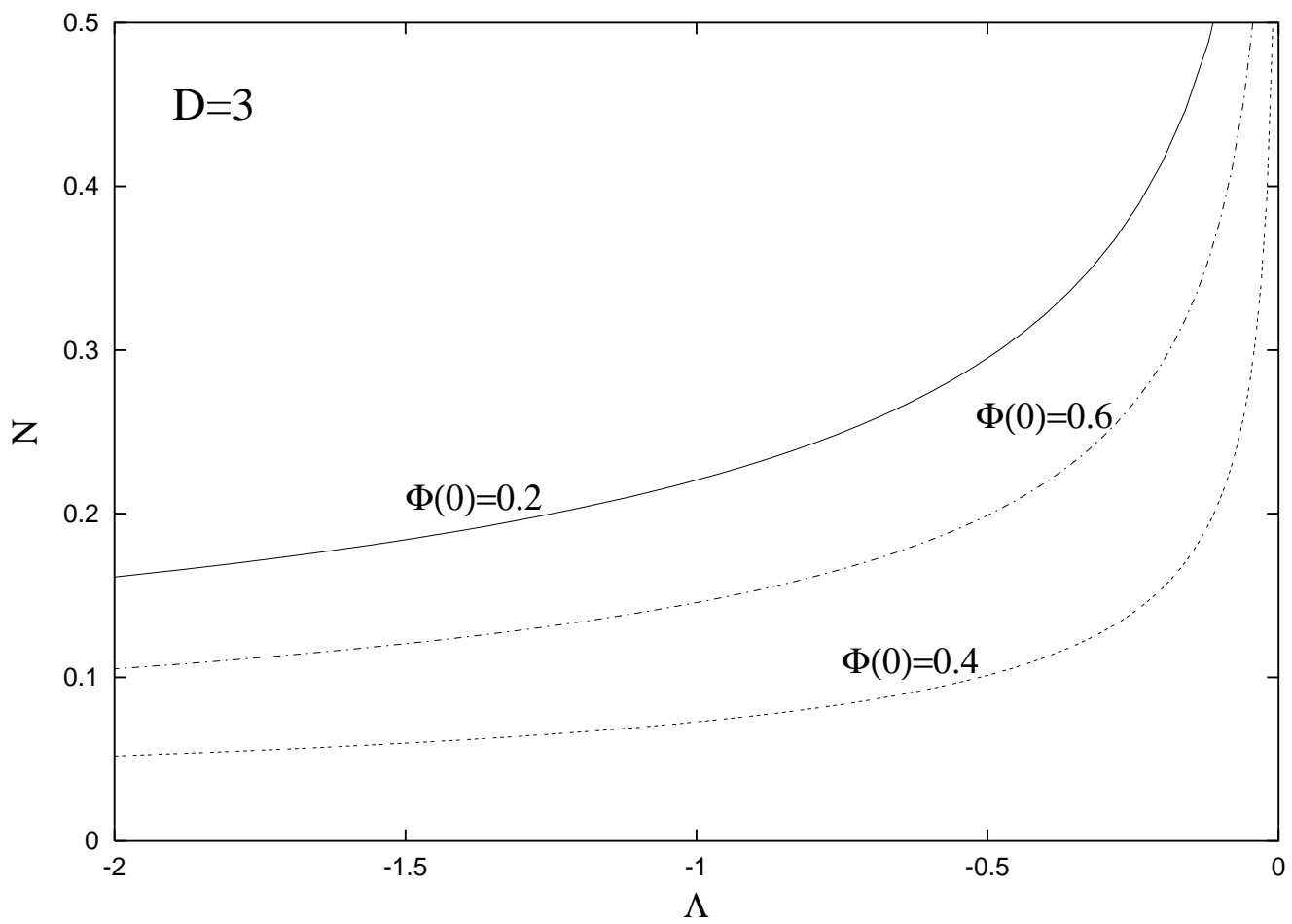


Figure 3a.

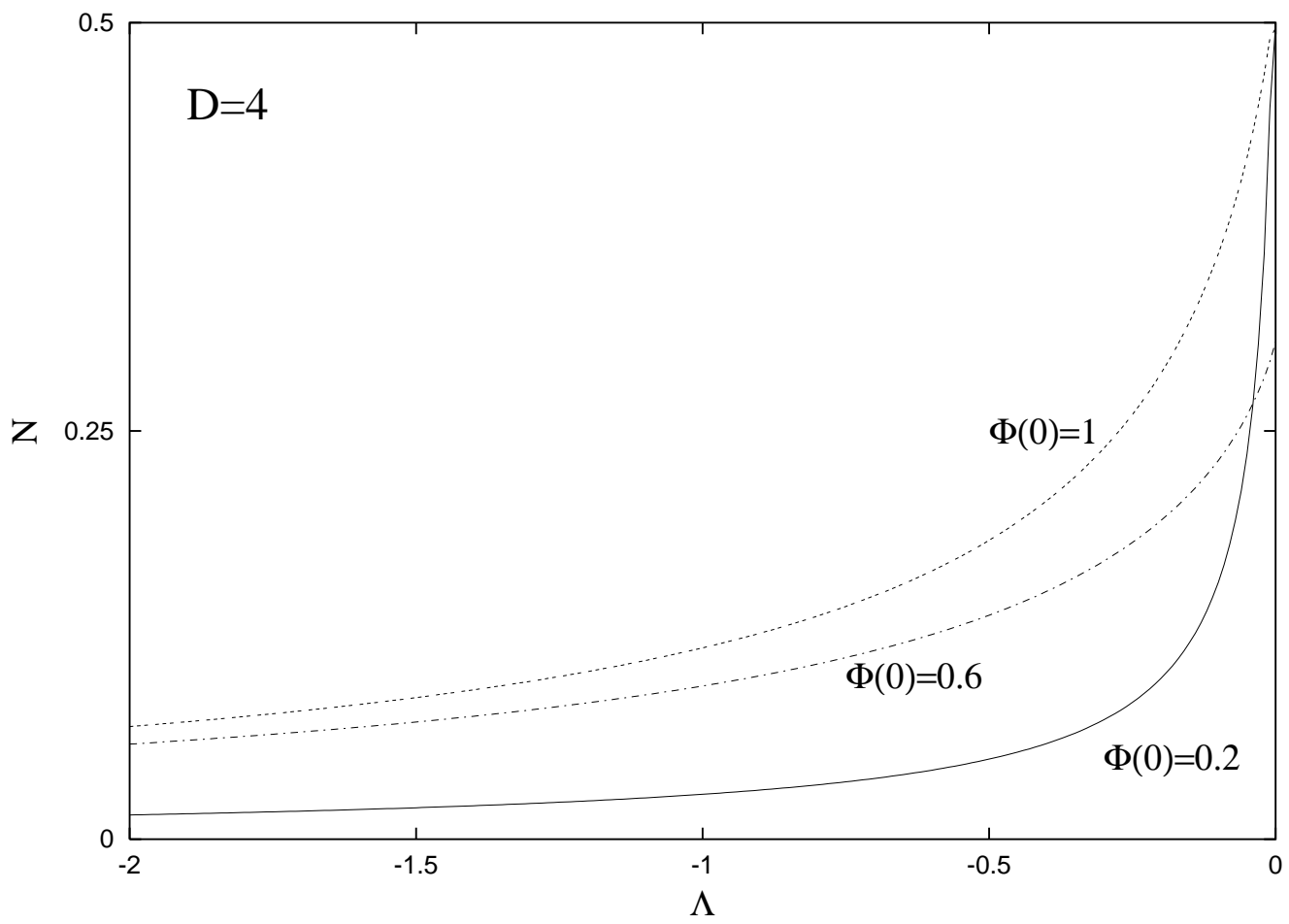


Figure 3b.

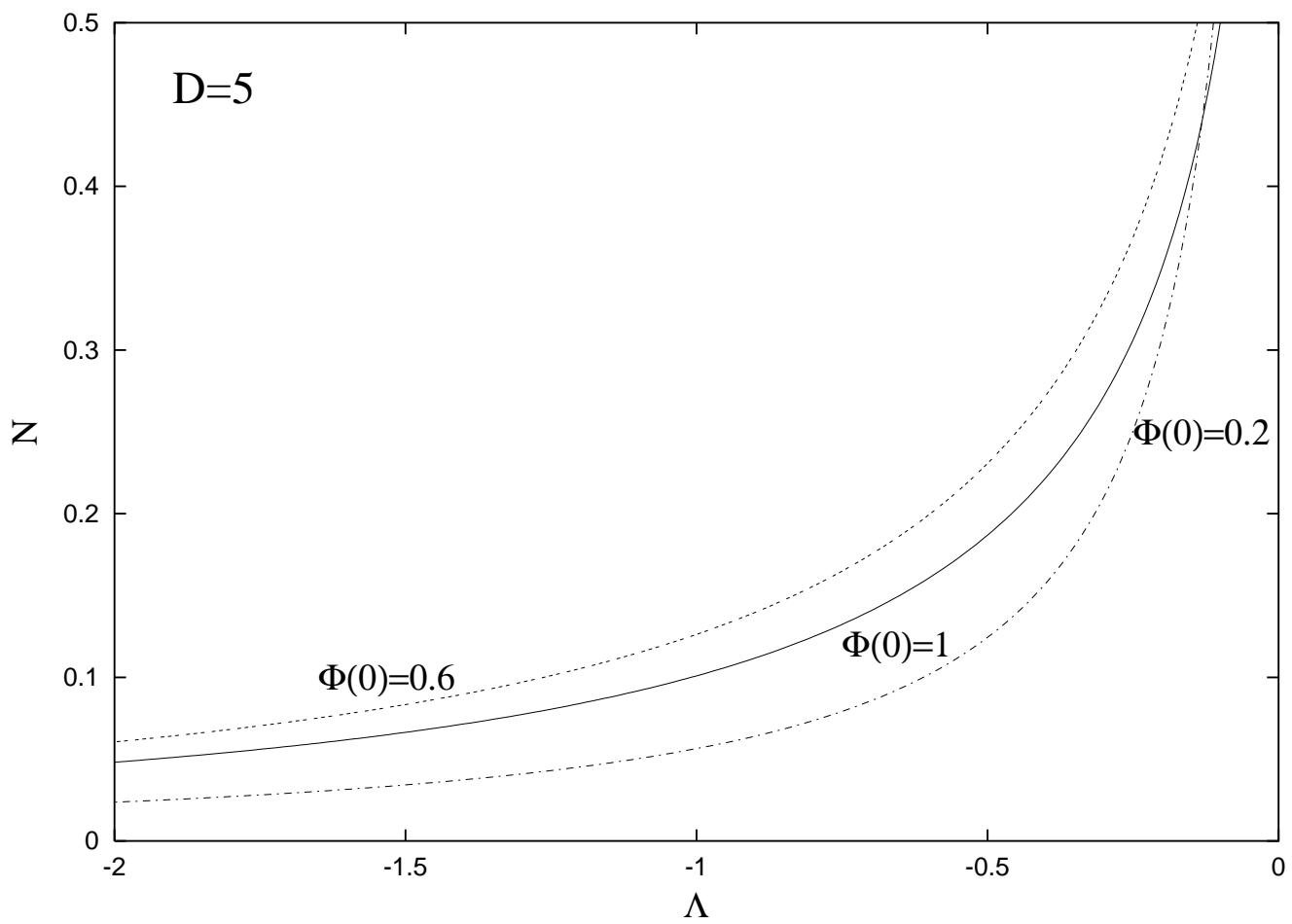


Figure 3c.

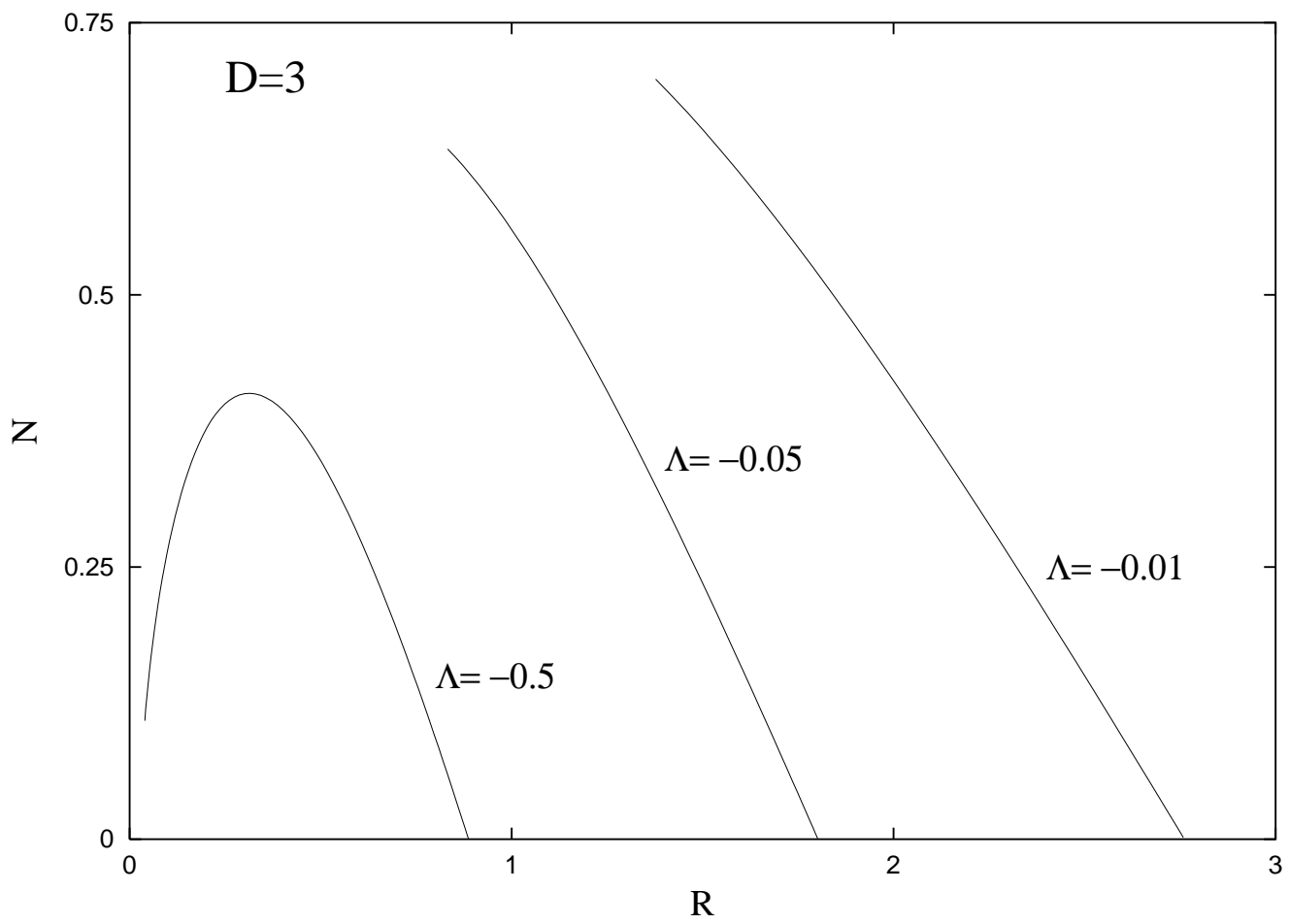


Figure 4a.

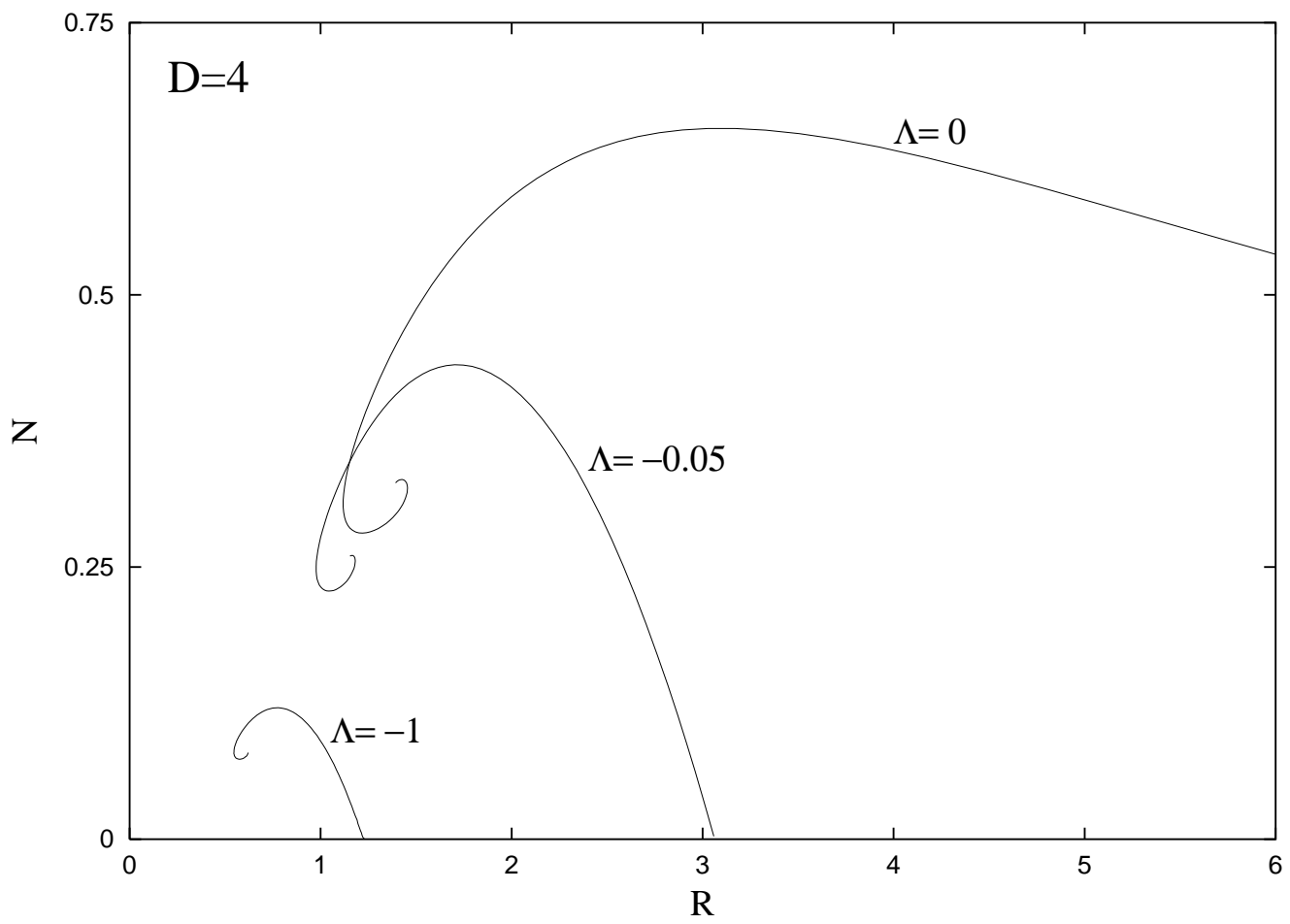


Figure 4b.

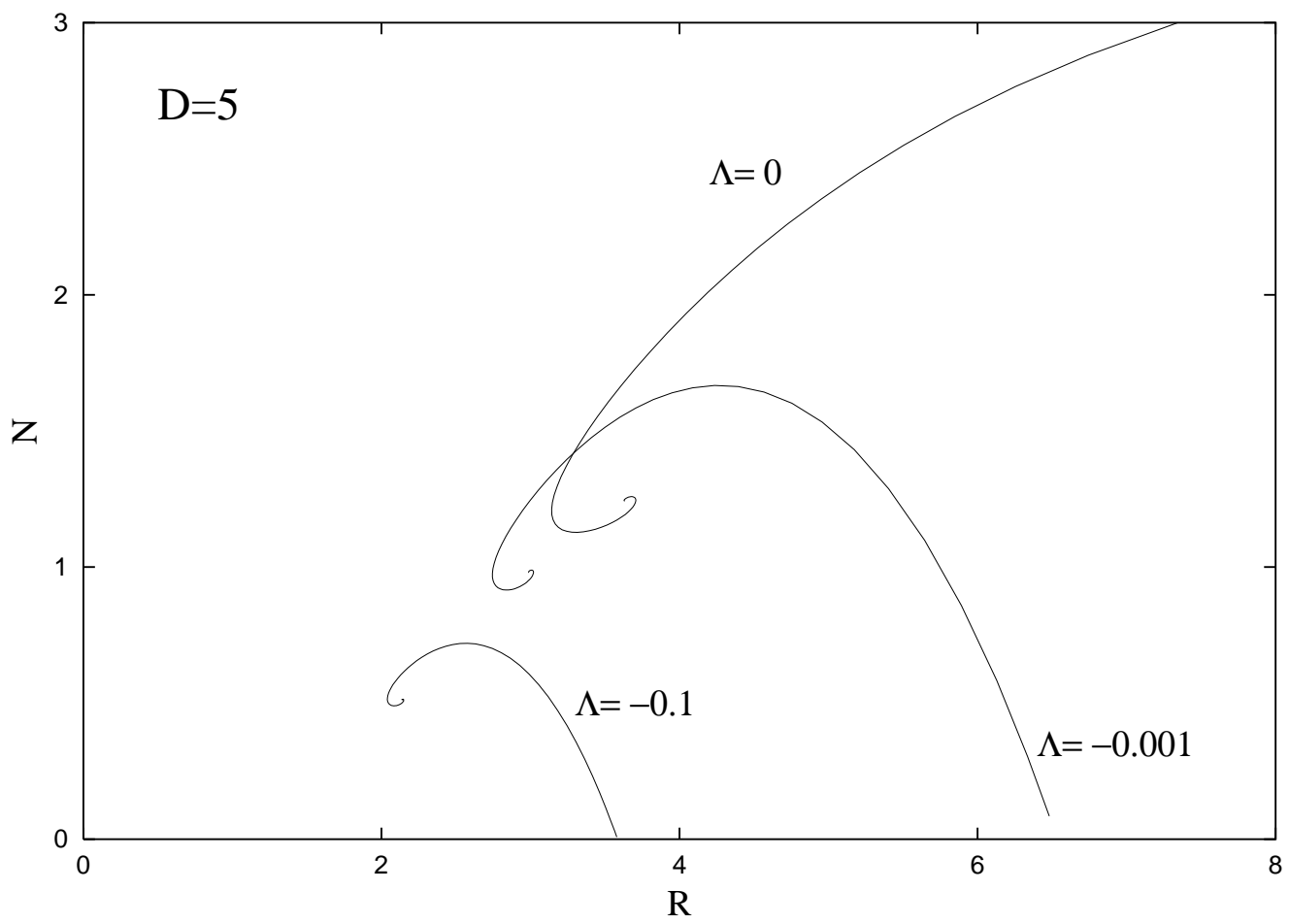


Figure 4c.

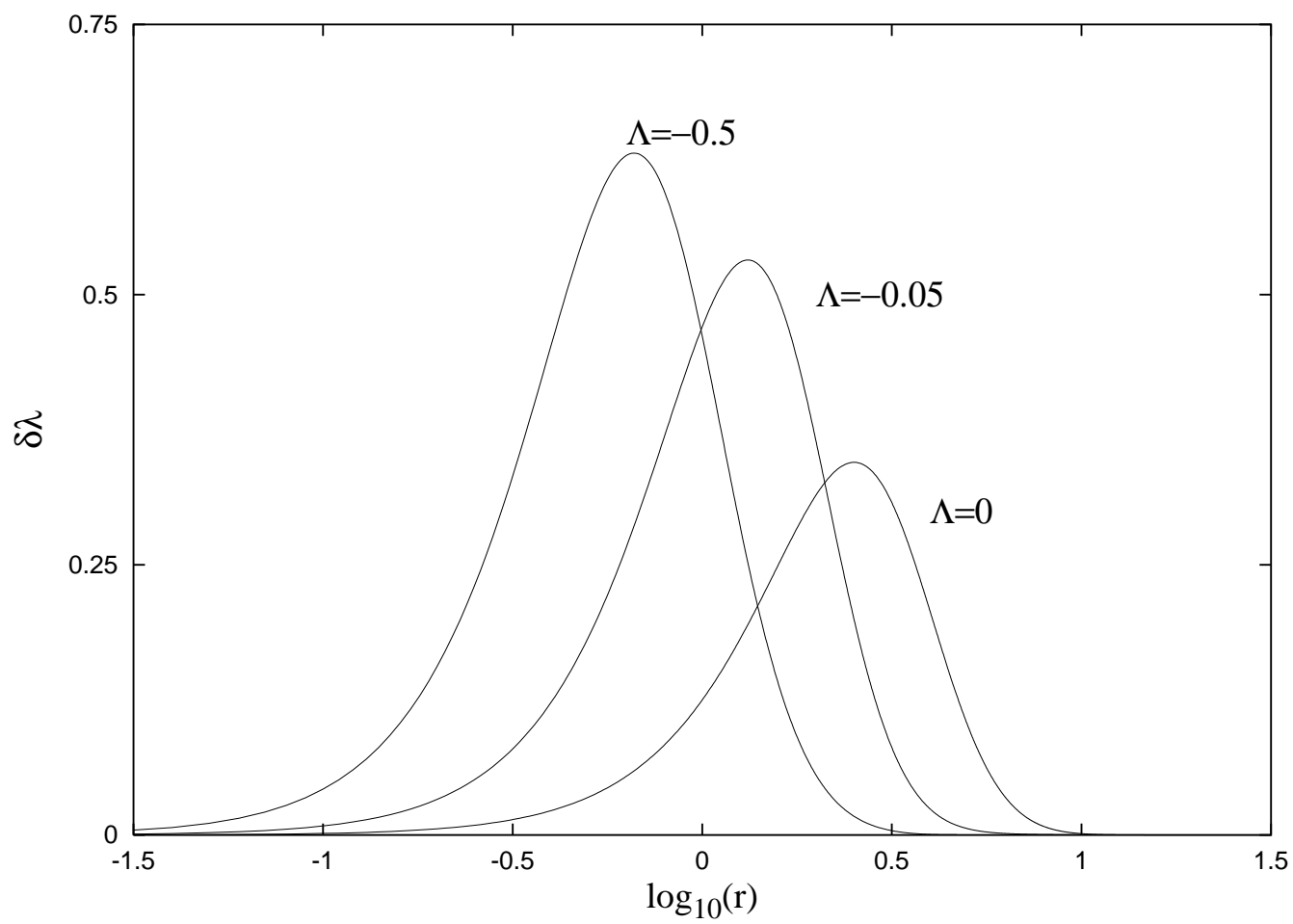


Figure 5.

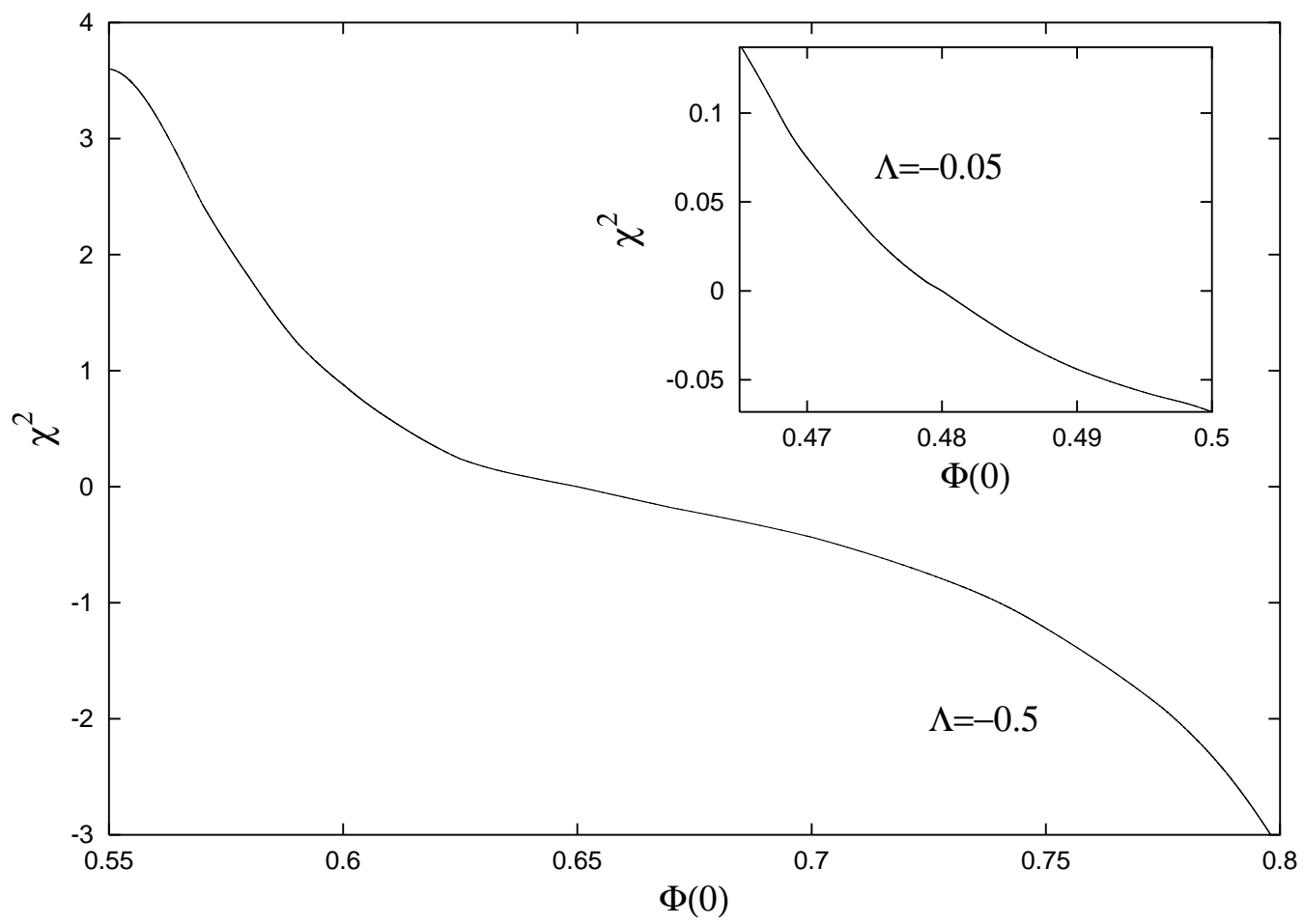


Figure 6.